PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Asymptotic Expansion of Solutions of the Differential Equation u<latex>$^{\text{iv}}$</latex>+<latex>$\lambda ^{2}$</latex>[(z<latex>^{2}</latex>+c)u<latex>$^{\prime \prime}$</latex>+azu<latex>$^{\prime}$</latex>+bu] = 0 for Large <latex>$|z|$</latex>

R. B. Paris and A. D. Wood

Phil. Trans. R. Soc. Lond. A 1980 **293**, 511-533
doi: 10.1098/rsta.1980.0002

THE ASYMPTOTIC EXPANSION OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $u^{\mathrm{iv}}+\lambda^2[(z^2+c)\,u''+azu'+bu]=0$ FOR LARGE $|z|$

By R. B. PARIS† and A. D. WOOD‡

† *Association Euratom-CEA sur la Fusion, Département de Physique du Plasma et de la Fusion Contrôlée, Centre d'Etudes Nucléaires,* 92260 *Fontenay-aux-Roses, France*

‡ *Department of Mathematics, Cranfield Institute of Technology, Cranfield, Bedford MK*43 0*AL, U.K.*

(*Communicated by K. G. Budden, F.R.S. – Received* 9 *November* 1978)

CONTENTS

The asymptotic expansion of solutions of the fourth order differential equation $u^{\mathrm{iv}}+\lambda^2[(z^2+c)\,u''+azu'+bu]=0$ are investigated for $|z|\rightarrow\infty$ where the parameters a, b, c and λ are supposed to be arbitrary complex constants with $\lambda, b\neq 0$. Exact solutions in the form of Laplace and Mellin–Barnes integrals, involving a Whittaker function and a Gauss hypergeometric function respectively, are used to define a fundamental system of solutions. The asymptotic expansion of these solutions is obtained in a full neighbourhood of the point at infinity and their asymptotic character is found to be either exponentially large or algebraic in certain sectors of the z-plane. The expansions corresponding to certain special values of the parameters a and b which yield logarithmic expansions are also treated. Linear combinations of these fundamental solutions which possess an exponentially small expansion for $|z|\rightarrow\infty$ in a certain sector are discussed.

1. Introduction and summary of the main results

In this paper we determine the asymptotic expansions of solutions of the fourth order differential equation

$$\frac{\mathrm{d}^4u}{\mathrm{d}z^4}+\lambda^2\left[(z^2+c)\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+az\frac{\mathrm{d}u}{\mathrm{d}z}+bu\right]=0 \tag{1.1}$$

for large values of the complex variable z. It is supposed that the parameters a, b, c and λ are arbitrary complex constants and that both λ and b are non-zero. The interest in an investigation of (1.1) lies in the fact that it is the simplest form of fourth order equation possessing two simple transition points (corresponding to the zeros of the coefficient of the second derivative in the neighbourhood of which the classical Liouville–Green (W.K.B.J.) approximations break down) for which exact solutions may be found in the form of integrals of the Laplace and Mellin–Barnes type. These integral representations are subsequently amenable to asymptotic analysis by standard techniques and enable asymptotic expansions of solutions of (1.1) to be found which are valid in a full neighbourhood of the point at infinity in the z-plane.

Previous investigations of (1.1) in the special case $c = 0$ have been carried out by Wood (1968, 1971) in connection with the deficiency indices of certain self-adjoint fourth order ordinary differential operators and by Paris (1975) in the discussion of the propagation of electromagnetic waves in an ionised gas permeated by an inhomogeneous magnetic field (Fidone & Paris 1974). The case with $a = 0$ has been studied by Erokhin (1971) employing methods different from those described in the present article. The differential equation (1.1) is a particular case of a more general fourth order equation, in which the coefficients are arbitrary analytic functions of z and λ, considered by Langer (1957, 1959) and Lin & Rabenstein (1960) in connection with the well known Orr–Sommerfeld equation describing the hydrodynamic stability of parallel shear flows. These authors were concerned with the determination of uniformly valid asymptotic expansions of solutions when $|\lambda|$ assumes large values for z lying in a domain containing one simple transition point only. A similar problem has been discussed by Lakin (1974) and Lakin & Nachman (1978), who encountered an equation of the form (1.1) describing the transverse oscillations of a rapidly rotating slender rod.

If the independent variable z is replaced by $\rho\lambda^{\frac{1}{2}}z$, where ρ denotes a root of the equation $\rho^4+1 = 0$, the canonical form of (1.1) may be taken without loss of generality to be the equation

$$\frac{d^4u}{dz^4}-\left[(z^2+4\kappa)\frac{d^2u}{dz^2}+az\frac{du}{dz}+bu\right] = 0 \tag{1.2}$$

where $\kappa = \frac{1}{4}\rho^2\lambda c$. We denote in § 2 a fundamental solution of (1.2) by the integral function $U_{\kappa,\nu}(a,z)$ where $\nu = \frac{1}{4}\{(a-1)^2-4b\}^{\frac{1}{2}}$, in terms of which a linearly independent set of solutions may be formed. The definition of $U_{\kappa,\nu}(a,z)$ is expressed by means of Laplace and Mellin–Barnes integrals, together with their analytic continuations, and its asymptotic expansion for $|z|\to\infty$ is determined in §§ 3–5 for arbitrary complex values of a, b (and hence ν) and κ by examination of these integral representations. The asymptotic character of $U_{\kappa,\nu}(a,z)$ is found to be either exponentially large or algebraic in different sectors of the z-plane and the discontinuous change occurring in the coefficients multiplying the exponential and algebraic expansions (the Stokes phenomenon) as certain rays in the z-plane are crossed is obtained. In many physical applications it is desirable to have solutions which possess an exponentially small expansion valid in some sector for $|z|\to\infty$. Such solutions are investigated in § 6 by consideration of linear combinations of the fundamental solutions of the type $U_{\kappa,\nu}(a,z)$.

We now summarize the principal results contained in this paper concerning the asymptotic expansion of the solution $U_{\kappa,\nu}(a,z)$ for $|z|\to\infty$. We define the formal asymptotic sums

$$S^{(1)}_{\kappa,\nu}(z) = (2\pi)^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(2z^2)^{-n}}{n!}\left(\tfrac{5}{2}-\tfrac{1}{2}a-2\kappa\right)_{2n}\,{}_3F_2\left(\begin{matrix}-n,\ \frac{1}{2}+\nu-\kappa,\ \frac{1}{2}-\nu-\kappa;\\ \frac{5}{4}-\frac{1}{4}a-\kappa,\ \frac{7}{4}-\frac{1}{4}a-\kappa;\end{matrix}\ \frac{1}{2}\right), \tag{1.3}$$

$$S^{(2)}_{\kappa,\nu}(z) = \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)} \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} \frac{(\frac{1}{2}-\nu-\kappa)_n}{(1-2\nu)_n} \Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+2n)\, {}_2F_1(-n, -n+2\nu; \tfrac{1}{2}+\nu+\kappa-n; \tfrac{1}{2}) \tag{1.4}$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} z^{-2n} \frac{\Gamma(2\nu-n)\,\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2n)}{\Gamma(\frac{1}{2}+\nu-\kappa-n)}\, {}_2F_1(-n, -n+2\nu; \tfrac{1}{2}+\nu-\kappa-n; \tfrac{1}{2}). \tag{1.5}$$

Provided that 2ν is not an integer or zero the asymptotic expansion of $U_{\kappa,\nu}(a,z)$ as $|z|\to\infty$ is

$$\begin{aligned} U_{\kappa,\nu}(a,z) &\simeq z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} S^{(1)}_{\kappa,\nu}(z) \quad (\arg z = 0) \\ &\simeq z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} S^{(1)}_{\kappa,\nu}(z) + (e^{-\pi i}z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} S^{(2)}_{\kappa,\nu}(z) + (e^{-\pi i}z)^{\frac{1}{2}-\frac{1}{2}a-2\nu} S^{(2)}_{\kappa,-\nu}(z) \quad (0 < \arg z < \tfrac{1}{2}\pi) \\ &\simeq z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} S^{(1)}_{\kappa,\nu}(z) + (e^{\pi i}z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} S^{(2)}_{\kappa,\nu}(z) + (e^{\pi i}z)^{\frac{1}{2}-\frac{1}{2}a-2\nu} S^{(2)}_{\kappa,-\nu}(z) \quad (-\tfrac{1}{2}\pi < \arg z < 0) \\ &\simeq (-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} S^{(2)}_{\kappa,\nu}(z) + (-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu} S^{(2)}_{\kappa,-\nu}(z) \quad (|\arg(-z)| < \tfrac{3}{4}\pi). \end{aligned} \tag{1.6}$$

When 2ν is a positive integer M or zero (it being sufficient to consider only positive M since $U_{\kappa,\nu}(a,z)$ is even in ν) the sums $S^{(2)}_{\kappa,\nu}(z)$ and $S^{(2)}_{\kappa,-\nu}(z)$ in (1.6) are to be replaced respectively by $S^{(3)}_{\kappa,\nu}(z)$ and $S^{(4)}_{\kappa,\nu}(z)$, where

$$S^{(3)}_{\kappa,\nu}(z) = \sum_{n=0}^{M-1} \frac{(-)^n}{n!} z^{-2n} \frac{\Gamma(2\nu-n)\,\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2n)}{\Gamma(\frac{1}{2}+\nu-\kappa-n)}\, {}_2F_1(-n, 2\nu-n; \tfrac{1}{2}+\nu-\kappa-n; \tfrac{1}{2}), \tag{1.7}$$

$$\begin{aligned} S^{(4)}_{\kappa,\nu}(z) = (-)^{2\nu} \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!(n+2\nu)!} \frac{\Gamma(\frac{1}{2}a-\frac{1}{2}+2\nu+2n)}{\Gamma(\frac{1}{2}-\nu-\kappa-n)} \Big[\{\psi(n+1)+\psi(n+2\nu+1) \\ -2\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n) + 2\ln(-\sqrt{2}z)\}\, {}_2F_1(-n, -n-2\nu; \tfrac{1}{2}-\nu-\kappa-n; \tfrac{1}{2}) \\ -2^{\kappa-\nu-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}+\nu-\kappa)_m (\frac{1}{2}-\nu-\kappa)_m}{2^{m+n} m!\, (\frac{1}{2}-\nu-\kappa-n)_m} \psi(\tfrac{1}{2}-\nu-\kappa+m-n) \Big]. \end{aligned} \tag{1.8}$$

When $\kappa = 0$ the asymptotic sums appearing in (1.6) simplify considerably and we have

$$S^{(1)}_{0,\nu}(z) = (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2z^2)^{-n}}{n!} (\tfrac{5}{2}-\tfrac{1}{2}a)_{2n}\, {}_3F_2\left(\begin{matrix}-n, \frac{1}{2}+\nu, \frac{1}{2}-\nu; \\ \frac{5}{4}-\frac{1}{4}a, \frac{7}{4}-\frac{1}{4}a;\end{matrix}\ \tfrac{1}{2}\right), \tag{1.9}$$

$$S^{(2)}_{0,\nu}(z) = \frac{2^{2\nu-1}\pi^{\frac{1}{2}}}{\sin\pi\nu} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{n!} \frac{\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+4n)}{\Gamma(n-\nu+1)} \tag{1.10}$$

and when ν is a positive integer M or zero

$$S^{(3)}_{0,\nu}(z) = 2^{2\nu-1} \sum_{n=0}^{M-1} \frac{(-)^n}{n!} (2z)^{-4n}\, \Gamma(\nu-n)\, \Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+4n), \tag{1.11}$$

$$\begin{aligned} S^{(4)}_{0,\nu}(z) = 2^{1-2\nu}(-)^{\nu} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{n!(n+\nu)!} \Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n)\, \{\tfrac{1}{4}\psi(n+1)+\tfrac{1}{4}\psi(n+\nu+1) \\ -\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n) + \ln(-2z)\}. \end{aligned} \tag{1.12}$$

The sums $S^{(3)}_{\kappa,\nu}(z)$ and $S^{(3)}_{0,\nu}(z)$ in (1.7) and (1.11) are to be interpreted as zero when $\nu = 0$.

The above expansions hold for all finite values of the parameters a, ν and κ as $|z|\to\infty$ except when $\frac{1}{2}a-\frac{1}{2}\pm 2\nu = -m$, $m = 0, 1, 2, \ldots$ where $U_{\kappa,\nu}(a,z)$ possesses poles. In § 2 it is shown that by suitably normalizing $U_{\kappa,\nu}(a,z)$ it is possible to define a solution of (1.2) which is analytic for all values of the parameters and which reduces to a polynomial in z of degree m at these exceptional points.

2. Discussion of the solutions and integral representations

We consider the solutions of the differential equation expressed in the canonical form

$$\frac{d^4u}{dz^4}-\left[(z^2+4\kappa)\frac{d^2u}{dz^2}+az\frac{du}{dz}+bu\right]=0 \tag{2.1}$$

for large values of $|z|$, where a, b and κ are arbitrary complex constants and where it will be supposed that b is non-zero. A solution of (2.1) in the form of a Laplace integral may be written as (Paris 1975)

$$U_{\kappa,\nu}(a,z)=\int_0^\infty e^{zt}\,t^{\frac{1}{2}a-\frac{5}{2}}\,W_{\kappa,\nu}(t^2)\,dt, \tag{2.2}$$

where $W_{\kappa,\nu}(t^2)$ is the Whittaker function of the second kind and the parameter ν is defined by

$$\nu=\tfrac{1}{4}\{(a-1)^2-4b\}^{\frac{1}{2}}. \tag{2.3}$$

The path of integration in (2.2) is taken to be the positive real axis but may pass to infinity anywhere in the sector $|\arg t|<\frac{1}{4}\pi$ since the Whittaker function behaves like $t^{2\kappa}\exp(-\frac{1}{2}t^2)$ as $|t|\to\infty$ in $|\arg t|<\frac{3}{4}\pi$ (see (4.18)). The representation (2.2) defines $U_{\kappa,\nu}(a,z)$ for all values of $\arg z$ and represents a continuous function of z uniformly convergent throughout the finite z-plane for

$$\operatorname{Re}(a)>1+4|\operatorname{Re}(\nu)|. \tag{2.4}$$

From the fact that $W_{\kappa,\nu}(t^2)$ is an even function of ν it follows that

$$U_{\kappa,\nu}(a,z)=U_{\kappa,-\nu}(a,z), \tag{2.5}$$

and it will be sufficient in the subsequent discussion to consider only values of ν with $\operatorname{Re}(\nu)\geqslant 0$.

An integral representation of $U_{\kappa,\nu}(a,z)$ of the Mellin–Barnes type may be obtained from (2.2) by employing the result

$$e^x=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(-s)\,(-x)^s\,ds \qquad |\arg(-x)|<\tfrac{1}{2}\pi,$$

in which the path of integration is suitably indented at $s=0$ so as to lie entirely to the left of all the poles of $\Gamma(-s)$. From (2.2) we then find

$$U_{\kappa,\nu}(a,z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(-s)\,(-z)^s\left\{\int_0^\infty t^{s+\frac{1}{2}a-\frac{5}{2}}\,W_{\kappa,\nu}(t^2)\,dt\right\}ds,$$

where the reversal of the order of integration is justified provided $|\arg(-z)|<\frac{1}{2}\pi$ and $\operatorname{Re}(a)>1+4|\operatorname{Re}(\nu)|$. Evaluation of the inner integral (Slater 1960, p. 53) then yields the Mellin–Barnes representation

$$U_{\kappa,\nu}(a,z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}g(s)\,\Gamma(-s)\,(-z)^s\,ds, \tag{2.6}$$

with

$$g(s)=\frac{1}{2}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}+\nu)\,\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}-\nu)}{\Gamma(\frac{1}{2}s+\frac{1}{4}a+\frac{1}{4}-\kappa)}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}+\nu,\ \frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}-\nu;\\ \frac{1}{2}s+\frac{1}{4}a+\frac{1}{4}-\kappa;\end{matrix}\ \frac{1}{2}\right) \tag{2.7}$$

$$=2^{\frac{1}{4}a-\frac{7}{4}+\kappa+\frac{1}{2}s}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}+\nu)\,\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}-\nu)}{\Gamma(\frac{1}{2}s+\frac{1}{4}a+\frac{1}{4}-\kappa)}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}+\nu-\kappa,\ \frac{1}{2}-\nu-\kappa;\\ \frac{1}{2}s+\frac{1}{4}a+\frac{1}{4}-\kappa;\end{matrix}\ \frac{1}{2}\right), \tag{2.8}$$

where we have used the Euler linear transformation for the Gauss hypergeometric function (Slater 1966, p. 10). The path of integration in (2.6) passes to the left of the poles of $\Gamma(-s)$

situated at $s = 0, 1, 2, \ldots$ but to the right of those of $\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}\pm\nu)$ situated at $s = \frac{1}{2}-\frac{1}{2}a\mp 2\nu -2m, m = 0, 1, 2, \ldots$. The function $g(s)$ is regular at the points $s = -\frac{1}{2}-\frac{1}{2}a-2\kappa-2m$ ($m = 0, 1, 2, \ldots$) corresponding to the poles of ${}_2F_1(\frac{1}{2})$, owing to the presence of $\Gamma(\frac{1}{2}s+\frac{1}{4}a+\frac{1}{4}-\kappa)$ in the denominator of $g(s)$.

For $|s|\to\infty$ the hypergeometric function in (2.8) behaves like $1+O(s^{-1})$ so that employing the asymptotic representation of the gamma function (Whittaker & Watson 1965, p. 251) the logarithm of the integrand in (2.6) for large $|s|$ in $|\arg s| < \pi$ is

$$(\tfrac{1}{4}a-\tfrac{7}{4}+\kappa-\tfrac{1}{2}s)\ln s-\tfrac{1}{2}s\ln 2+\tfrac{1}{2}s-\ln(\sin\pi s)+s\ln(-\sqrt{2}z)+O(1). \tag{2.9}$$

The dominant real part of (2.9) as $|s|\to\infty$ on the imaginary s-axis is consequently

$$\mathrm{Re}\,(\tfrac{1}{4}a-\tfrac{7}{4}+\kappa)\ln|s|-\Delta|s|+O(1),\ \Delta = \tfrac{3}{4}\pi\pm\arg(-z).$$

The integral representation (2.6) has been proved in $|\arg(-z)| < \frac{1}{2}\pi$, but since the modulus of the integrand as $|s|\to\infty$ on the path of integration is $O(|s|^{\mathrm{Re}(\frac{1}{4}a-\frac{7}{4}+\kappa)}\,\mathrm{e}^{-\Delta|s|})$, it follows by the principle of analytic continuation that (2.6) represents $U_{\kappa,\nu}(a,z)$ in the wider domain $|\arg(-z)| < \frac{3}{4}\pi$.

The condition (2.4) is equivalent to restricting the poles of $g(s)$ at the points $s = \frac{1}{2}-\frac{1}{2}a\pm 2\nu -2m$ ($m = 0, 1, 2, \ldots$) to lie entirely in $\mathrm{Re}\,(s) < 0$. This restriction on the validity of the Mellin–Barnes representation (2.6) may be removed to include values of the parameters a and ν not satisfying (2.4) by suitably deforming the path of integration in (2.6) into $\mathrm{Re}\,(s)\geqslant 0$ to pass to the right of any of the poles of $g(s)$ which lie in $\mathrm{Re}\,(s)\geqslant 0$ but to the left of the poles of $\Gamma(-s)$. This separation of the poles is possible for all values of the parameters except when

$$a = 1\pm 4\nu-2m,\quad m = 0, 1, 2, \ldots, \tag{2.10}$$

corresponding to the coincidence of poles of $g(s)$ with those of $\Gamma(-s)$.

From (2.2) or (2.6) it can be shown that $U_{\kappa,\nu}(a,z)$ possesses the Maclaurin expansion (Paris 1975)

$$U_{\kappa,\nu}(a,z) = \sum_{n=0}^{\infty} g(n)\frac{z^n}{n!}, \tag{2.11}$$

where $g(n)$ is defined in (2.7) or (2.8) with $s = n$. The function $U_{\kappa,\nu}(a,z)$ is then seen to be an integral function of z of order 2 (Copson 1935, p. 178) uniformly and absolutely convergent throughout the finite z-plane for all values of a, ν and κ except those in (2.10), where it possesses poles. The expansion (2.11) represents the analytic continuation of the integral representations (2.2) and (2.6) to all values of $\arg z$ and, with the exception of (2.10), all values of the parameters. From (2.11) and (2.8) it follows that derivatives of $U_{\kappa,\nu}(a,z)$ with respect to z are given by

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}U_{\kappa,\nu}(a,z) = U_{\kappa,\nu}(a+2p,z),\quad p = 1, 2, \ldots.$$

An alternative form of expansion of $U_{\kappa,\nu}(a,z)$ in terms of the Hermite polynomials $H_n(z)$ may be found by expressing (2.11) together with (2.7) as a double sum which, after suitable rearrangement, may be summed diagonally to give

$$U_{\kappa,\nu}(a,z) = \sum_{n=0}^{\infty}\frac{(-)^n}{(2n)!}\frac{\Gamma(\frac{1}{4}a-\frac{1}{4}+\nu+n)\,\Gamma(\frac{1}{4}a-\frac{1}{4}-\nu+n)}{2^{n+1}\Gamma(\frac{1}{4}a+\frac{1}{4}+\kappa+n)}H_{2n}\left(\frac{\mathrm{i}z}{\sqrt{2}}\right)$$
$$-\mathrm{i}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)!}\frac{\Gamma(\frac{1}{4}a+\frac{1}{4}+\nu+n)\,\Gamma(\frac{1}{4}a+\frac{1}{4}-\nu+n)}{2^{n+\frac{3}{2}}\Gamma(\frac{1}{4}a+\frac{3}{4}+\kappa+n)}H_{2n+1}\left(\frac{\mathrm{i}z}{\sqrt{2}}\right). \tag{2.12}$$

The expansion (2.12) has the merit of exhibiting $U_{\kappa,\nu}(a,z)$ as the sum of an even and odd series of Hermite polynomials which, apart from (2.10), converge uniformly and absolutely for all finite values of z and the parameters a, ν and κ.

In the special case $\kappa = 0$ the Whittaker function in (2.2) reduces to the modified Bessel function of the second kind since $W_{0,\nu}(t^2) = \pi^{-\frac{1}{2}} t K_\nu(\frac{1}{2}t^2)$, so that

$$U_{0,\nu}(a,z) = \pi^{-\frac{1}{2}}\int_0^\infty e^{zt}\, t^{\frac{1}{2}a-\frac{3}{2}} K_\nu(\tfrac{1}{2}t^2)\, dt \qquad \mathrm{Re}\,(a) > 1+4|\mathrm{Re}\,(\nu)|, \tag{2.13}$$

(Wood 1968, 1971; Paris 1975). The hypergeometric function appearing in (2.8) is summable by Gauss's second theorem (Slater 1966, p. 243) and we consequently obtain the representations

$$U_{0,\nu}(a,z) = \frac{2^{\frac{1}{2}a-\frac{9}{2}}}{\pi i}\pi^{-\frac{1}{2}}\int_{-\infty i}^{\infty i}(-2z)^s\,\Gamma(-s)\,\Gamma(\tfrac{1}{4}s+\tfrac{1}{8}a-\tfrac{1}{8}+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{4}s+\tfrac{1}{8}a-\tfrac{1}{8}-\tfrac{1}{2}\nu)\,ds \qquad |\arg(-z)| < \tfrac{3}{4}\pi \tag{2.14}$$

$$= 2^{\frac{1}{2}a-\frac{7}{2}}\pi^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(2z)^n}{n!}\Gamma(\tfrac{1}{4}n+\tfrac{1}{8}a-\tfrac{1}{8}+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{4}n+\tfrac{1}{8}a-\tfrac{1}{8}-\tfrac{1}{2}\nu), \tag{2.15}$$

where, as in (2.6), the path of integration separates the poles of $\Gamma(-s)$ from those of $\Gamma(\frac{1}{4}s+\frac{1}{8}a-\frac{1}{8}\pm\frac{1}{2}\nu)$. Both representations (2.14) and (2.15) hold for general values of a and ν with the exception of those in (2.10). Solutions of the differential equation (2.1) in this case have been shown to be expressible as ${}_2F_3$ generalized hypergeometric functions and are discussed further in connection with a more general differential equation of order n ($\geqslant 3$) in Paris (1978). For certain values of the parameters a and b these solutions reduce to Bessel and related functions (Paris 1975).

When $a = 3$ the function $U_{\kappa,\nu}(a,z)$ may be represented in terms of products of Weber functions (Paris 1975). This latter result in the particular case $\kappa = 0$ may be obtained directly from (2.13) since when $a = 3$ it is of the form of an integral discussed by Meijer (1935). A further special case arises when $\kappa = \frac{1}{2}\pm\nu$ since the Whittaker function $W_{\kappa,\nu}(t^2)$ reduces to $t^{1\pm 2\nu}\exp(-\frac{1}{2}t^2)$ and the resulting integral (2.2) may be evaluated as a Weber parabolic cylinder function

$$U_{\frac{1}{2}\pm\nu,\nu}(a,z) = \Gamma(\tfrac{1}{2}a-\tfrac{1}{2}\pm 2\nu)\,e^{\frac{1}{4}z^2}D_{\frac{1}{2}-\frac{1}{2}a\mp 2\nu}(-z). \tag{2.16}$$

The solutions of (2.1) considered by Erokhin (1971) in the case $a = 0$ involved integrals of the Weber parabolic cylinder function of the form

$$\int s^{\frac{1}{2}a-\frac{3}{2}-2\nu}(1-s^2)^{\nu-\kappa-\frac{1}{2}}(1+s^2)^{\nu+\kappa-\frac{1}{2}}\,e^{\frac{1}{4}z^2s^2}D_{\frac{1}{2}-\frac{1}{2}a-2\nu}(zs)\,ds,$$

taken along appropriately chosen paths in the complex s-plane. It may be readily shown that the solution $U_{\kappa,\nu}(a,z)$ may be expressed as an integral of the above type over the interval [0, 1] by employing the representation of the Whittaker function (Slater 1960, p. 51; Whittaker & Watson 1965, p. 340)

$$W_{\kappa,\nu}(t^2) = \frac{t^{1+2\nu}e^{-\frac{1}{2}t^2}}{\Gamma(\frac{1}{2}+\nu-\kappa)}\int_0^\infty e^{-ut^2}u^{\nu-\kappa-\frac{1}{2}}(1+u)^{\nu+\kappa-\frac{1}{2}}\,du \qquad |\arg t| < \tfrac{1}{2}\pi, \quad \mathrm{Re}\,(\tfrac{1}{2}+\nu-\kappa) > 0,$$

in (2.2). Provided $\mathrm{Re}\,(a) > 1+4|\mathrm{Re}\,(\nu)|$ and $\mathrm{Re}\,(\frac{1}{2}+\nu-\kappa) > 0$, so that the order of integration may be reversed, we find

$$U_{\kappa,\nu}(a,-z) = \frac{2^{1-2\nu}}{\Gamma(\frac{1}{2}+\nu-\kappa)}\int_1^\infty w(w^2-1)^{\nu-\kappa-\frac{1}{2}}(w^2+1)^{\nu+\kappa-\frac{1}{2}}\left\{\int_0^\infty e^{zt-\frac{1}{2}w^2t^2}\,t^{\frac{1}{2}a-\frac{3}{2}+2\nu}\,dt\right\}dw$$

$$= 2^{1-2\nu}\frac{\Gamma(\frac{1}{2}a-\frac{1}{2}+2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)}\int_0^1 s^{\frac{1}{2}a-\frac{3}{2}-2\nu}(1-s^2)^{\nu-\kappa-\frac{1}{2}}(1+s^2)^{\nu+\kappa-\frac{1}{2}}\,e^{\frac{1}{4}z^2s^2}D_{\frac{1}{2}-\frac{1}{2}a-2\nu}(zs)\,ds, \tag{2.17}$$

upon evaluating the inner integral in terms of a parabolic cylinder function and making the transformation $w = 1/s$ in the outer integral. It will be observed that in (2.17) the parameter κ is no longer present in the transcendental function and appears in the integrand only in the algebraic factor $[(1+s^2)/(1-s^2)]^\kappa$. The integral representations discussed by Erokhin were taken along paths which avoided the origin $s = 0$, with end points the branch points $s = \pm 1$, $\pm \mathrm{i}$ and the point at infinity in suitably chosen directions.

The differential equation (2.1) is unaltered if we simultaneously replace z and κ by ωz and $\omega^2\kappa$, where ω denotes a fourth root of unity. It follows that, except for the special values of the parameters in (2.10), a fundamental system of solutions of (2.1) is represented by

$$U_{\kappa,\nu}(a,\ z), \quad U_{-\kappa,\nu}(a,\ \mathrm{i}z), \quad U_{\kappa,\nu}(a,\ -z), \quad U_{-\kappa,\nu}(a,\ -\mathrm{i}z). \tag{2.18}$$

From the transformation in (1.2) a system of solutions of the general equation (1.1) may consequently be expressed in terms of $U_{\kappa,\nu}(a,z)$ by

$$U_{\frac{1}{4}\mathrm{i}c\lambda,\nu}(a,\ \lambda^{\frac{1}{2}}z\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}), \quad U_{-\frac{1}{4}\mathrm{i}c\lambda,\nu}(a,\ \lambda^{\frac{1}{2}}z\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}), \quad U_{-\frac{1}{4}\mathrm{i}c\lambda,\nu}(a,\ \lambda^{\frac{1}{2}}z\,\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}}), \quad U_{\frac{1}{4}\mathrm{i}c\lambda,\nu}(a,\ \lambda^{\frac{1}{2}}z\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}}). \tag{2.19}$$

Since the solution $U_{\kappa,\nu}(a,z)$ possesses poles for values of a and ν given by (2.10), a more suitable form for the representation of a fundamental system of solutions of (2.1) may be taken to be the function

$$\mathfrak{U}_{\kappa,\nu}(a,\ z) = \frac{U_{\kappa,\nu}(a,\ z)}{\Gamma(\frac{1}{2}a-\frac{1}{2}+2\nu)\,\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu)}, \tag{2.20}$$

which is analytic for all values of the parameters a, ν and κ. For the special values of a and ν in (2.10), however, the set of solutions $\mathfrak{U}_{\kappa,\nu}(a,z)$, $\mathfrak{U}_{-\kappa,\nu}(a,\mathrm{i}z)$, $\mathfrak{U}_{\kappa,\nu}(a,-z)$ and $\mathfrak{U}_{-\kappa,\nu}(a,-\mathrm{i}z)$ is no longer linearly independent as may be seen by inspection of (2.11) and (2.8). For when $\frac{1}{4}a-\frac{1}{4}\pm\nu = -\frac{1}{2}m$, $m = 1, 2, \ldots$, we have† $(\kappa \neq 0)$

$$\left.\begin{aligned} \mathfrak{U}_{\kappa,\nu}(a,\ z) &= (-)^m\,\mathfrak{U}_{\kappa,\nu}(a,\ -z), \\ \mathfrak{U}_{-\kappa,\nu}(a,\ \mathrm{i}z) &= (-)^m\,\mathfrak{U}_{-\kappa,\nu}(a,\ -\mathrm{i}z), \end{aligned}\right\} \tag{2.21}$$

where $\mathfrak{U}_{\kappa,\nu}(a,z)$ and $\mathfrak{U}_{-\kappa,\nu}(a,\mathrm{i}z)$ are either even or odd polynomials in z of degree m according as m is respectively even or odd. Because $\mathfrak{U}_{\kappa,\nu}(a,z)$ is even in ν we need only consider $\nu = \frac{1}{4}a-\frac{1}{4}+\frac{1}{2}m$, $m = 1, 2, \ldots$, so that a second solution may be taken in the form (cf. the definition of the Bessel function of the second kind in Watson (1944), p. 57 *et seq.*)

$$\frac{\mathfrak{U}_{\kappa,\nu}(a,\ z)\cos(2\nu-\frac{1}{2}a+\frac{1}{2})\,\pi-\mathfrak{U}_{\kappa,\nu}(a,\ -z)}{\Gamma(1+2\nu-\frac{1}{2}a+\frac{1}{2})\sin(2\nu-\frac{1}{2}a+\frac{1}{2})\,\pi}, \tag{2.22}$$

which represents an independent solution of (2.1) for *all* values of the parameters a, ν and κ. When $2\nu-\frac{1}{2}a+\frac{1}{2}$ takes on positive integral values m, (2.22) possesses the limit

$$\frac{1}{2\pi m!}\left[\frac{\partial\mathfrak{U}_{\kappa,\nu}(a,\ z)}{\partial\nu}-(-)^m\,\frac{\partial\mathfrak{U}_{\kappa,\nu}(a,\ -z)}{\partial\nu}\right]$$

evaluated at $\nu = \frac{1}{4}a-\frac{1}{4}+\frac{1}{2}m$. In a similar manner a fourth linearly independent solution may be defined as

$$\frac{\mathfrak{U}_{-\kappa,\nu}(a,\ \mathrm{i}z)\cos(2\nu-\frac{1}{2}a+\frac{1}{2})\,\pi-\mathfrak{U}_{-\kappa,\nu}(a,\ -\mathrm{i}z)}{\Gamma(1+2\nu-\frac{1}{2}a+\frac{1}{2})\sin(2\nu-\frac{1}{2}a+\frac{1}{2})\,\pi}, \tag{2.23}$$

so that (2.22) and (2.23) together with $\mathfrak{U}_{\kappa,\nu}(a,z)$ and $\mathfrak{U}_{-\kappa,\nu}(a,\mathrm{i}z)$ constitute a fundamental system of solutions of (2.1) for all values of the parameters.

† The case $m = 0$ is degenerate since this corresponds to $b = 0$.

3. The asymptotic expansion of $U_{\kappa,\nu}(a, z)$ in $|\arg(-z)| < \frac{3}{4}\pi$

The asymptotic expansion of the solution $U_{\kappa,\nu}(a,z)$ for $|z| \to \infty$ in the domain $|\arg(-z)| < \frac{3}{4}\pi$ will be obtained by means of the Mellin–Barnes integral representation (2.6)

$$U_{\kappa,\nu}(a,z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} g(s)\,\Gamma(-s)\,(-z)^s\,ds, \quad |\arg(-z)| < \tfrac{3}{4}\pi, \tag{3.1}$$

where $g(s)$ is defined in (2.8). The path of integration is suitably indented along the imaginary axis so as to separate the poles of $\Gamma(-s)$ from those of $g(s)$ situated at

$$s = \tfrac{1}{2} - \tfrac{1}{2}a \pm 2\nu - 2n, \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

and represents the function $U_{\kappa,\nu}(a,z)$ for arbitrary κ and all values of a and ν except those given in (2.10). The two infinite sequences of poles in (3.2) are simple except when ν is zero, when they form a single sequence of double poles, and when 2ν is a positive integer M (cf. (2.5)) with all but the first M poles being double.

In the manner described by Slater (1966, p. 143) we consider the integral (3.1) taken round the rectangular contour with vertices at $\pm i\xi$, $-T \pm i\xi$ where ξ and T are positive with $\xi > \frac{1}{2}|\mathrm{Im}\,(a)| + 2|\mathrm{Im}\,(\nu)|$, in order that the sequences of poles (3.2) lie between the upper and lower sides of the rectangle, and T such that none of the poles (3.2) lies on the line joining the vertices $-T \pm i\xi$. The contributions from the upper and lower sides given by $s = -x \pm i\xi$, $0 \leqslant x \leqslant T$ vanish as ξ is made to tend to infinity if $|\arg(-z)| < \frac{3}{4}\pi$ since the modulus of the integrand on these paths can be seen from (2.9) to be $O\{|z|^{-x}\xi^{\mathrm{Re}(\frac{1}{4}a - \frac{7}{4} + \kappa + \frac{1}{2}x)}\,e^{-\Delta\xi}\}$ with $\Delta = \frac{3}{4}\pi \pm \arg(-z)$. The modulus of the integrand on the path $s = -T + it$ parallel to the imaginary axis is $|z|^{-T} O\{|t|^{\mathrm{Re}(\frac{1}{4}a - \frac{7}{4} + \kappa + \frac{1}{2}T)}\,e^{-\Delta|t|}\}$ where the constant implied in the O symbol is independent of z. It follows that the contribution from the side joining the vertices $-T \pm i\xi$ is $O(|z|^{-T})$ for $|z| \to \infty$ in $|\arg(-z)| < \frac{3}{4}\pi$.

When 2ν is non-integral the residues of the integrand in (3.1) at the simple poles (3.2) are

$$\begin{aligned} &2^{\kappa \pm \nu - n - \frac{1}{2}} \frac{(-)^n}{n!} \frac{\Gamma(\pm 2\nu - n)\,\Gamma(\frac{1}{2}a - \frac{1}{2} \mp 2\nu + 2n)}{\Gamma(\frac{1}{2} \pm \nu - \kappa - n)}\, {}_2F_1\left(\begin{matrix}\frac{1}{2} + \nu - \kappa,\ \frac{1}{2} - \nu - \kappa; \\ \frac{1}{2} \pm \nu - \kappa - n;\end{matrix}\ \tfrac{1}{2}\right)(-z)^{\frac{1}{2} - \frac{1}{2}a \pm 2\nu - 2n} \\ &= \frac{(-)^n}{n!} \frac{\Gamma(\pm 2\nu - n)\,\Gamma(\frac{1}{2}a - \frac{1}{2} \mp 2\nu + 2n)}{\Gamma(\frac{1}{2} \pm \nu - \kappa - n)}\, {}_2F_1(-n, -n \pm 2\nu; \tfrac{1}{2} \pm \nu - \kappa - n; \tfrac{1}{2})\,(-z)^{\frac{1}{2} - \frac{1}{2}a \pm 2\nu - 2n}, \end{aligned} \tag{3.3}$$

by Euler's transformation. Owing to the presence of $-n$ in the numeratorial parameter the ${}_2F_1$ hypergeometric function in (3.3) is a finite series. A further transformation of (3.3) may be obtained by making use of the result (Abramowitz & Stegun 1965, p. 559, formula 15.3.6)

$${}_2F_1(-n, -n \pm 2\nu; \tfrac{1}{2} \pm \nu - \kappa - n; \tfrac{1}{2}) = (-)^n \frac{(\frac{1}{2} \mp \nu - \kappa)_n}{(\frac{1}{2} \mp \nu + \kappa)_n}\, {}_2F_1(-n, -n \pm 2\nu; \tfrac{1}{2} \pm \nu + \kappa - n; \tfrac{1}{2}), \tag{3.4}$$

where $(\alpha)_n$ is Pochhammer's notation for $\Gamma(\alpha + n)/\Gamma(\alpha)$. Employing the properties of the gamma function and (3.4) we find

$$\begin{aligned} &(-)^n \frac{\Gamma(\pm 2\nu - n)}{\Gamma(\frac{1}{2} \pm \nu - \kappa - n)}\, {}_2F_1(-n, -n \pm 2\nu; \tfrac{1}{2} \pm \nu - \kappa - n; \tfrac{1}{2}) \\ &\qquad = \frac{\Gamma(\pm 2\nu)}{\Gamma(\frac{1}{2} \pm \nu - \kappa)} \frac{(\frac{1}{2} \mp \nu - \kappa)_n}{(1 \mp 2\nu)_n}\, {}_2F_1(-n, -n \pm 2\nu; \tfrac{1}{2} \pm \nu + \kappa - n; \tfrac{1}{2}), \end{aligned} \tag{3.5}$$

so that an alternative expression for the residues (3.3) is

$$\frac{1}{n!}\frac{\Gamma(\pm 2\nu)}{\Gamma(\frac{1}{2}\pm\nu-\kappa)}\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}\mp 2\nu+2n)\,\frac{(\frac{1}{2}\mp\nu-\kappa)_n}{(1\mp 2\nu)_n}\,{}_2F_1(-n,\ -n\pm 2\nu;\ \tfrac{1}{2}\pm\nu+\kappa-n;\ \tfrac{1}{2})\,(-z)^{\frac{1}{2}-\frac{1}{2}a\pm 2\nu-2n}. \tag{3.6}$$

Application of Cauchy's theorem then yields the asymptotic expansion for $|z|\to\infty$ in $|\arg(-z)| < \frac{3}{4}\pi$

$$\begin{aligned} U_{\kappa,\nu}(a,z) \simeq{}& \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu}\sum_{n=0}^{\infty}\frac{z^{-2n}}{n!}\frac{(\frac{1}{2}-\nu-\kappa)_n}{(1-2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+2n)\\ &\times{}_2F_1\left(\begin{matrix}-n,\ -n+2\nu;\\ \frac{1}{2}+\nu+\kappa-n;\end{matrix}\ \tfrac{1}{2}\right)+\frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu}\sum_{n=0}^{\infty}\frac{z^{-2n}}{n!}\\ &\times\frac{(\frac{1}{2}+\nu-\kappa)_n}{(1+2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n)\,{}_2F_1\left(\begin{matrix}-n,\ -n-2\nu;\\ \frac{1}{2}-\nu+\kappa-n;\end{matrix}\ \tfrac{1}{2}\right), \end{aligned} \tag{3.7}$$

which holds for all values of the parameters except those in (2.10) and when 2ν is an integer or zero.

When 2ν is a positive integer M or zero there are M simple poles of $g(s)$ situated at $s=\frac{1}{2}-\frac{1}{2}a+2\nu-2n$, $n=0,1,\ldots,M-1$ with the residues

$$\frac{(-)^n}{n!}\frac{\Gamma(2\nu-n)\,\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2n)}{\Gamma(\frac{1}{2}+\nu-\kappa-n)}\,{}_2F_1(-n,\ 2\nu-n;\ \tfrac{1}{2}+\nu-\kappa-n;\ \tfrac{1}{2})\,(-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu-2n},$$

together with an infinite sequence of double poles at $s=\frac{1}{2}-\frac{1}{2}a-2\nu-2n$, $n=0,1,2,\ldots$. The residues at the double poles may be determined by consideration of the logarithmic derivate of the function

$$\begin{aligned} y(s) &= 4(\tfrac{1}{2}s+\tfrac{1}{4}a-\tfrac{1}{4}+\nu+n)^2\,\Gamma(-s)(-z)^s g(s)\\ &= \frac{4\Gamma(-s)(-z)^s g(s)}{\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}+\nu)\,\Gamma(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}-\nu)}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{4}a+\frac{3}{4}+\nu+n)}{(\frac{1}{2}s+\frac{1}{4}a-\frac{5}{4}+\nu+n)\ldots(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}+\nu)}\\ &\quad\times\frac{\Gamma(\frac{1}{2}s+\frac{1}{4}a+\frac{3}{4}+\nu+n)}{(\frac{1}{2}s+\frac{1}{4}a-\frac{5}{4}+\nu+n)\ldots(\frac{1}{2}s+\frac{1}{4}a-\frac{1}{4}-\nu)}, \end{aligned}$$

evaluated at $s=\frac{1}{2}-\frac{1}{2}a-2\nu-2n$. By employing the fact that the derivate with respect to s of the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma+\frac{1}{2}s;\frac{1}{2})$, where α, β and γ are independent of s, is given by

$$\tfrac{1}{2}\psi(\gamma+\tfrac{1}{2}s)\,{}_2F_1(\alpha,\ \beta;\ \gamma+\tfrac{1}{2}s;\ \tfrac{1}{2})-\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma+\frac{1}{2}s)_m}\frac{2^{-m}}{m!}\psi(\gamma+m+\tfrac{1}{2}s),$$

the residues at the double poles are found to be

$$\begin{aligned} &\frac{(-)^{2\nu}\Gamma(\frac{1}{2}a-\frac{1}{2}+2\nu+2n)}{n!(n+2\nu)!\,\Gamma(\frac{1}{2}-\nu-\kappa-n)}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu-2n}\Big[\{\psi(n+1)+\psi(n+2\nu+1)-2\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n)\\ &\qquad+2\ln(-\sqrt{2z})\}\,{}_2F_1(-n,\ -n-2\nu;\ \tfrac{1}{2}-\nu-\kappa-n;\ \tfrac{1}{2})-2^{\kappa-\nu-\frac{1}{2}}\\ &\qquad\times\sum_{m=0}^{\infty}\frac{(\frac{1}{2}+\nu-\kappa)_m(\frac{1}{2}-\nu-\kappa)_m}{2^{m+n}m!(\frac{1}{2}-\nu-\kappa-n)_m}\psi(\tfrac{1}{2}-\nu-\kappa+m-n)\Big]. \end{aligned}$$

The asymptotic expansion of $U_{\kappa,\nu}(a,z)$ when $2\nu = M$ for $|z| \to \infty$ in $|\arg(-z)| < \frac{3}{4}\pi$ is consequently

$$U_{\kappa,\nu}(a,z) \simeq (-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} \sum_{n=0}^{M-1} (-)^n \frac{z^{-2n}}{n!} \frac{\Gamma(2\nu-n)\,\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2n)}{\Gamma(\frac{1}{2}+\nu-\kappa-n)}$$
$$\times {}_2F_1(-n, 2\nu-n; \tfrac{1}{2}+\nu-\kappa-n; \tfrac{1}{2}) + (-)^{2\nu}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu}$$
$$\times \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!(n+2\nu)!} \frac{\Gamma(\frac{1}{2}a-\frac{1}{2}+2\nu+2n)}{\Gamma(\frac{1}{2}-\nu-\kappa-n)} \Big[\{\psi(n+1)+\psi(n+2\nu+1)-2\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n)$$
$$+2\ln(-\sqrt{2z})\}\,{}_2F_1(-n, -n-2\nu; \tfrac{1}{2}-\nu-\kappa-n; \tfrac{1}{2}) - 2^{\kappa-\nu-\frac{1}{2}}$$
$$\times \sum_{m=0}^{\infty} \frac{(\frac{1}{2}+\nu-\kappa)_m\,(\frac{1}{2}-\nu-\kappa)_m}{2^{m+n}\,m!(\frac{1}{2}-\nu-\kappa-n)_m}\,\psi(\tfrac{1}{2}-\nu-\kappa+m-n)\Big], \tag{3.8}$$

where when $\nu = 0$, the first sum in (3.8) is to be interpreted as zero.

For $\kappa = 0$ the hypergeometric functions in (3.7) are summable by Gauss's second theorem so that after some manipulation of the gamma functions we obtain the expansion (Wood 1968; Paris 1975)

$$U_{0,\nu}(a,z) \simeq \frac{\pi^{\frac{1}{2}}(-z)^{\frac{1}{2}-\frac{1}{2}a}}{2\sin\pi\nu}\Big\{(-2z)^{2\nu} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{n!} \frac{\Gamma(4n+\frac{1}{2}a-\frac{1}{2}-2\nu)}{\Gamma(n-\nu+1)}$$
$$-(-2z)^{-2\nu} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{n!} \frac{\Gamma(4n+\frac{1}{2}a-\frac{1}{2}+2\nu)}{\Gamma(n+\nu+1)}\Big\}, \tag{3.9}$$

for $|z| \to \infty$ in $|\arg(-z)| < \frac{3}{4}\pi$ valid for all values of a and ν (non-integral) except those in (2.10). The expansion for $\kappa = 0$ and ν integral is more readily obtained by considering the integral (2.14). When ν takes on integral values M ($\geqslant 0$) the integrand of (2.14) possesses M simple poles at $s = \frac{1}{2}-\frac{1}{2}a+2\nu-4n$, $n = 0, 1, \ldots, M-1$ of residue

$$2^{\frac{1}{2}a-\frac{3}{2}}\,\pi^{-\frac{1}{2}} \frac{(-)^n}{n!} (-2z)^{\frac{1}{2}-\frac{1}{2}a+2\nu-4n}\,\Gamma(\nu-n)\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+4n),$$

and an infinite sequence of double poles at $s = \frac{1}{2}-\frac{1}{2}a-2\nu-4n$, $n = 0, 1, 2, \ldots$ of residue

$$2^{\frac{1}{2}a+\frac{1}{2}} \frac{\pi^{-\frac{1}{2}}(-)^{\nu}}{n!(n+\nu)!} (-2z)^{\frac{1}{2}-\frac{1}{2}a-2\nu-4n}\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n)\,\{\tfrac{1}{4}\psi(n+1)$$
$$+\tfrac{1}{4}\psi(n+\nu+1)-\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n)+\ln(-2z)\}.$$

The asymptotic expansion of $U_{0,\nu}(a,z)$ for $|z| \to \infty$ in $|\arg(-z)| < \frac{3}{4}\pi$ when $\nu = M$ is therefore

$$U_{0,\nu}(a,z) \simeq 2^{2\nu-1}\pi^{-\frac{1}{2}}(-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} \sum_{n=0}^{M-1} \frac{(-)^n}{n!} (2z)^{-4n}\,\Gamma(\nu-n)\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+4n)$$
$$+2^{1-2\nu}\,\pi^{-\frac{1}{2}}(-)^{\nu}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{n!(n+\nu)!}\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n)\,\{\tfrac{1}{4}\psi(n+1)$$
$$+\tfrac{1}{4}\psi(n+\nu+1)-\psi(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+4n)+\ln(-2z)\}. \tag{3.10}$$

In the special case $\nu = 0$ all the poles to the left of the path of integration in (2.14) are double and we consequently have the result

$$U_{0,0}(a,z) \simeq \pi^{-\frac{1}{2}}(-z)^{\frac{1}{2}-\frac{1}{2}a} \sum_{n=0}^{\infty} \frac{(2z)^{-4n}}{(n!)^2}\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+4n)\,\{\psi(n+1)-2\psi(\tfrac{1}{2}a-\tfrac{1}{2}+4n)+2\ln(-2z)\} \tag{3.11}$$

for $|z| \to \infty$ in $|\arg(-z)| < \frac{3}{4}\pi$ provided $a-1$ is not zero or a negative even integer.

4. The asymptotic expansion of $U_{\kappa,\nu}(a, z)$ in $|\arg z| < \frac{1}{2}\pi$

The asymptotic expansion of $U_{\kappa,\nu}(a, z)$ for $|z| \to \infty$ in the domain $|\arg z| < \frac{1}{2}\pi$ will be determined by consideration of the integral (2.2), in which the path of integration may be deformed to pass to infinity anywhere in the sector $|\arg t| < \frac{1}{4}\pi$. We write

$$U_{\kappa,\nu}(a, z) = \left\{\int_{C_1} + \int_{C_2}\right\} e^{zt} t^{\frac{1}{2}a-\frac{3}{2}} W_{\kappa,\nu}(t^2)\, dt, \quad \mathrm{Re}(a) > 1 + 4|\mathrm{Re}(\nu)|, \tag{4.1}$$

where as shown in figure 1 the path C_1 is the ray $\arg t = \pm\pi - \arg z$ emanating from the origin and terminating at the point $t = -\bar{z}$ (the bar denoting the complex conjugate), with the upper or lower sign being chosen according as z lies in the upper or lower half plane, and the path C_2 is a line parallel to the real t-axis commencing at $t = -\bar{z}$ and passing to infinity in $|\arg t| < \frac{1}{4}\pi$. The path C_2 passes through the point $t = z$, which can be seen to be a saddle point of the integrand in (4.1) for large $|z|$ in $|\arg z| < \frac{1}{2}\pi$ since $W_{\kappa,\nu}(t^2)$ behaves asymptotically like $t^{2\kappa} \exp(-\frac{1}{2}t^2)$ in $|\arg t| < \frac{3}{4}\pi$. In a right half plane containing $t = z$ the path C_2 is the path of steepest descent through the saddle point.

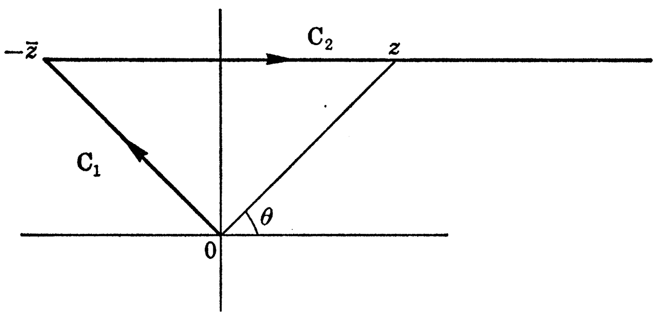

Figure 1. The paths C_1 and C_2 in the t-plane with $\theta = \arg z$ shown positive. Along the path C_1 $\arg t = \pi - \theta$ and the path C_2 passes to infinity parallel to the positive real axis through the saddle point at $t = z$.

The asymptotic behaviour of the Whittaker function in the left half t-plane may be found from the analytic continuation formula (Slater 1960, p. 28)

$$W_{\kappa,\nu}(t^2) = \pm\frac{2\pi i M_{\kappa,\nu}(t^2 e^{\mp 2\pi i})}{\Gamma(\frac{1}{2}-\nu-\kappa)\,\Gamma(1+2\nu)} - e^{\mp 2\pi i\nu} W_{\kappa,\nu}(t^2 e^{\mp 2\pi i}), \tag{4.2}$$

for $\frac{1}{2}\pi \leqslant |\arg t| \leqslant \pi$, where the upper or lower signs are to be taken according as t lies in the upper or lower half planes respectively. From the asymptotic behaviour of the Whittaker functions (Slater 1960, p. 61) we therefore have for $|t| \to \infty$ in $\frac{1}{2}\pi \leqslant |\arg t| \leqslant \pi$†

$$W_{\kappa,\nu}(t^2) \simeq t^{2\kappa} e^{-\frac{1}{2}t^2} \pm \frac{2\pi i\, e^{\pm 2\pi i\kappa}\, t^{-2\kappa}}{\Gamma(\frac{1}{2}+\nu-\kappa)\,\Gamma(\frac{1}{2}-\nu-\kappa)} e^{\frac{1}{2}t^2}. \tag{4.3}$$

Consequently $W_{\kappa,\nu}(t^2)$ tends to zero exponentially only in the sector $|\arg t| < \frac{1}{4}\pi$ and, in fact, use of the analytic continuation formulae to cover successive Riemann surfaces shows this to be the only sector. It follows that $W_{\kappa,\nu}(t^2)$ satisfies

$$|W_{\kappa,\nu}(t^2)| \leqslant K|t|^{2|\mathrm{Re}(\kappa)|} e^{\frac{1}{2}|t^2|\,|\cos 2\phi|}, \quad \phi = \arg t, \tag{4.4}$$

for $|t| \to \infty$ in $\frac{1}{4}\pi \leqslant |\arg t| \leqslant \pi$, where K is an assignable constant.

† This result is not in contradiction with the behaviour $t^{2\kappa} \exp(-\frac{1}{2}t^2)$ in $|\arg t| < \frac{3}{4}\pi$ since in their common domains of validity $\frac{3}{4}\pi \leqslant |\arg t| \leqslant \frac{1}{2}\pi$ the second term in (4.3) is subdominant. The rays $\arg t = \pm\frac{3}{4}\pi$ are anti-Stokes lines for the function $W_{\kappa,\nu}(t^2)$.

For non-integral values of 2ν, $W_{\kappa,\nu}(t^2)$ is defined as (Slater 1960, p. 14)

$$W_{\kappa,\nu}(t^2) = \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)} M_{\kappa,-\nu}(t^2) + \frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\kappa)} M_{\kappa,\nu}(t^2), \tag{4.5}$$

where the Whittaker function of the first kind $M_{\kappa,\nu}(t^2)$, is given by

$$M_{\kappa,\nu}(t^2) = \mathrm{e}^{-\frac{1}{2}t^2} t^{1+2\nu} {}_1F_1(\tfrac{1}{2}+\nu-\kappa;\ 1+2\nu;\ t^2), \tag{4.6}$$

with ${}_1F_1$ being the confluent hypergeometric function. Denoting by $A_n(\nu)$ the coefficient of t^{2n} in the expansion in ascending powers of t^2 of

$$\mathrm{e}^{-\frac{1}{2}t^2} {}_1F_1(\tfrac{1}{2}-\nu-\kappa;\ 1-2\nu;\ t^2),$$

we may write $W_{\kappa,\nu}(t^2)$ in the form

$$W_{\kappa,\nu}(t^2) = \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)} t^{1-2\nu} \sum_{n=0}^{\infty} A_n(\nu)\, t^{2n} + \frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\kappa)} t^{1+2\nu} \sum_{n=0}^{\infty} A_n(-\nu)\, t^{2n}, \tag{4.7}$$

provided 2ν is non-integral, where

$$\begin{aligned} A_n(\nu) &= \sum_{m=0}^{n} \frac{(-)^m}{m!} 2^{-m} \frac{(\frac{1}{2}-\nu-\kappa)_{n-m}}{(1-2\nu)_{n-m}} \frac{1}{(n-m)!} \\ &= \frac{1}{n!} \frac{(\frac{1}{2}-\nu-\kappa)_n}{(1-2\nu)_n} {}_2F_1(-n, -n+2\nu;\ \tfrac{1}{2}+\nu+\kappa-n;\ \tfrac{1}{2}), \end{aligned} \tag{4.8}$$

on making use of the result $(\alpha)_{n-m} = (-)^m(\alpha)_n/(1-\alpha-n)_m$.

(a) *Contribution from the path* C_1

In the subsequent discussion of the contribution arising from the path C_1 we present only the analysis for the first series in (4.7), since the contribution resulting from the second series may be obtained by replacing ν by $-\nu$. We define

$$\Psi_{N,\nu}(t) = t^{\frac{1}{2}a-\frac{5}{2}} M_{\kappa,-\nu}(t^2) - t^{\frac{1}{2}a-\frac{3}{2}-2\nu} \sum_{n=0}^{N-1} A_n(\nu)\, t^{2n}, \tag{4.9}$$

and write $z = x\,\mathrm{e}^{\mathrm{i}\theta}$, where x and θ will be supposed real. Then

$$\int_{C_1} \mathrm{e}^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}} M_{\kappa,-\nu}(t^2)\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\mu(\pm\pi-\theta)} \sum_{n=0}^{N-1} A_n(\nu)\, \mathrm{e}^{-2n\mathrm{i}\theta} \int_0^x \mathrm{e}^{-xu} u^{\mu-1+2n}\,\mathrm{d}u - \mathrm{e}^{-\mathrm{i}\theta} \int_0^x \mathrm{e}^{-xu} \Psi_{N,\nu}(t)\,\mathrm{d}u, \tag{4.10}$$

where $\mu = \frac{1}{2}a - \frac{1}{2} - 2\nu$ and on the path C_1 $t = u \exp \mathrm{i}(\pm\pi - \theta)$, with the upper or lower sign being chosen according as z lies in the upper or lower half plane respectively.

From the analytic continuation formula and the asymptotic behaviour of $M_{\kappa,\nu}(t^2)$ (Slater 1960, pp. 28, 61) we have $\Psi_{N,\nu}(t) = O\{|t|^{2|\mathrm{Re}(\kappa)|} \exp(\frac{1}{2}|t|^2|\cos 2\phi|)\}$ as $|t| \to \infty$ in $|\arg t| \leqslant \pi$ (cf. (4.4)). At the end point $t = -\bar{z}$ of the path C_1, $\mathrm{e}^{-xu}\Psi_{N,\nu}(t)$ is therefore dominated by $\exp\{-x^2 + \frac{1}{2}x^2|\cos 2\theta|\} \leqslant \exp(-\frac{1}{2}x^2)$. The last integral in (4.10) may consequently be estimated by Laplace's method (Olver 1974, p. 80) where the dominant contribution arises from the neighbourhood of the origin where $\Psi_{N,\nu}(t) = O(t^{\mu-1+2N})$. Then as $x \to +\infty$ we find

$$\left|\int_0^x \mathrm{e}^{-xu} \Psi_{N,\nu}(t)\,\mathrm{d}u\right| = O(x^{\frac{1}{2}-\frac{1}{2}a+2\nu-2N}). \tag{4.11}$$

Evaluation of the first integral on the right hand side of (4.10) yields

$$\int_0^x e^{-xu}\, u^{\mu-1+2n}\, du = x^{-\mu-2n}\{\Gamma(\mu+2n)-\Gamma(\mu+2n,\, x^2)\} \simeq x^{-\mu-2n}\,\Gamma(\mu+2n) \tag{4.12}$$

as $x \to +\infty$ since the incomplete gamma function $\Gamma(\mu+2n, x^2)$ possesses the asymptotic behaviour $x^{2\mu+4n-2}\, e^{-x^2}$ (Abramowitz & Stegun 1965, p. 263). From (4.10), (4.11) and (4.12) we consequently have for $x \to +\infty$

$$\int_{C_1} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{4}}\, M_{\kappa,-\nu}(t^2)\, dt = (e^{\mp\pi i}z)^{\frac{1}{2}-\frac{1}{2}a+2\nu}\left\{\sum_{n=0}^{N-1}\frac{A_n(\nu)}{z^{2n}}\,\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+2n)+O(z^{-2N})\right\}, \tag{4.13}$$

and the contribution to $U_{\kappa,\nu}(a,z)$ resulting from the path C_1 therefore has the asymptotic expansion

$$\frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu}\sum_{n=0}^{\infty}\frac{z^{-2n}}{n!}\frac{(\frac{1}{2}-\nu-\kappa)_n}{(1-2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+2n)\,{}_2F_1(-n,\,-n+2\nu;\tfrac{1}{2}+\tfrac{1}{2}\nu+\kappa-n;\tfrac{1}{2})$$
$$+\frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu}\sum_{n=0}^{\infty}\frac{z^{-2n}}{n!}\frac{(\frac{1}{2}+\nu-\kappa)_n}{(1+2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n)$$
$$\times\,{}_2F_1(-n,\,-n-2\nu;\,\tfrac{1}{2}-\nu+\kappa-n;\,\tfrac{1}{2}), \tag{4.14}$$

where $-z$ is to be interpreted as $e^{\mp\pi i}z$ according as z lies in the upper or lower half plane respectively. The algebraic expansion (4.14), which has been obtained for values of the parameters satisfying (2.4), may be seen to be the same as that in (3.7) valid in the domain $|\arg(-z)| < \frac{3}{4}\pi$ and for all values of a, ν and κ with the exception of (2.10) and integral values of 2ν (see § 5).

(b) *Contribution from the path* C_2

The integral along the path C_2 through the point $t = z$ may be written as

$$\left\{\int_{-\xi+i\eta}^{-\delta\xi+i\eta}+\int_{-\delta\xi+i\eta}^{\infty+i\eta}\right\} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{4}}\, W_{\kappa,\nu}(t^2)\, dt, \quad z = \xi+i\eta, \tag{4.15}$$

where ξ and η denote the real and imaginary parts of z and a δ is chosen to satisfy $0 < \delta \leqslant 1$ such that the end point at $-\delta\xi+i\eta$ lies in the sector $\frac{1}{2}\pi < |\arg t| < \frac{3}{4}\pi$. With this choice of δ the Whittaker function in the second integral in (4.15) may be replaced by its asymptotic expansion valid in $|\arg t| < \frac{3}{4}\pi$. It is to be noted that if $\frac{1}{4}\pi < |\arg z| < \frac{1}{2}\pi$ the end point $-\bar{z} = -\xi+i\eta$ already lies in $\frac{1}{2}\pi < |\arg t| < \frac{3}{4}\pi$ and so we may then put $\delta = 1$ and the first integral in (4.15) vanishes.

We parametrize the path of integration over the t-interval $[-\xi+i\eta,\, -\delta\xi+i\eta]$ in (4.15) by $t = -\alpha\xi+i\eta$, where $0 < \delta \leqslant \alpha \leqslant 1$ so that, for large $|z|$ in $|\arg z| \leqslant \frac{1}{4}\pi$,

$$I = \left|\int_{-\xi+i\eta}^{-\delta\xi+i\eta} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{4}}\, W_{\kappa,\nu}(t^2)\, dt\right| \leqslant \xi\, e^{\frac{1}{2}\pi\,|\mathrm{Im}\,(a)|}\int_\delta^1 e^{-\alpha\xi^2-\eta^2}\,(\alpha^2\xi^2+\eta^2)^{\frac{1}{4}\mathrm{Re}\,(a)-\frac{5}{8}}\,|W_{\kappa,\nu}(t^2)|\, d\alpha$$
$$\leqslant K\xi\, e^{\frac{1}{2}\pi\,|\mathrm{Im}\,(a)|}\int_\delta^1(\alpha^2\xi^2+\eta^2)^{\frac{1}{4}\mathrm{Re}\,(a)-\frac{5}{8}+|\mathrm{Re}\,(\kappa)|}\, e^{-F(\alpha)}\, d\alpha$$

by (4.4), where $\qquad F(\alpha) = \alpha\xi^2+\eta^2-\frac{1}{2}|\alpha^2\xi^2-\eta^2|.$

As the path of integration is described from $\alpha = 1$ to $\alpha = \delta$ $F(\alpha)$ steadily decreases from its value $x^2-\frac{1}{2}x^2|\cos 2\theta|$ at $\alpha = 1$ to the value $\frac{1}{2}\eta^2+\delta(1+\frac{1}{2}\delta)\,\xi^2$ at $\alpha = \delta$. Consequently, for $|z| > 1$

$$I \leqslant K\xi\, e^{\frac{1}{2}\pi|\mathrm{Im}\,(a)|}\, e^{-F(\delta)}\int_\delta^1(\alpha^2\xi^2+\eta^2)^{\frac{1}{4}\mathrm{Re}\,(a)-\frac{5}{8}+|\mathrm{Re}\,(\kappa)|}\, d\alpha$$
$$\leqslant K\xi(1-\delta)\, e^{\frac{1}{2}\pi|\mathrm{Im}\,(a)|}(|z|\,|\mathrm{cosec}\,\theta|)^{|\frac{1}{2}\mathrm{Re}\,(a)-\frac{5}{4}+2|\mathrm{Re}\,(\kappa)||}\, e^{-F(\delta)}, \tag{4.16}$$

where, provided $\arg z \neq 0$, $F(\delta) \to +\infty$ as $|z| \to \infty$ so that $I \to 0$ and the contribution to (4.15) from the interval $[-\xi+\mathrm{i}\eta, -\delta\xi+\mathrm{i}\eta]$ is asymptotically negligible as $|z| \to \infty$ in $0 < |\arg z| \leqslant \frac{1}{4}\pi$.

We now consider the integral

$$\int_{-\delta\xi+\mathrm{i}\eta}^{\infty+\mathrm{i}\eta} \mathrm{e}^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, W_{\kappa,\nu}(t^2)\,\mathrm{d}t, \tag{4.17}$$

where δ has been chosen so that the path of integration lies entirely in the domain $|\arg t| < \frac{3}{4}\pi$. Employing the asymptotic expansion of $W_{\kappa,\nu}(t^2)$ valid for arbitrary ν and κ (Slater 1960, p. 61)

$$W_{\kappa,\nu}(t^2) = \mathrm{e}^{-\frac{1}{2}t^2}\, t^{2\kappa} \sum_{n=0}^{N-1} \frac{(-)^n}{n!} (\tfrac{1}{2}+\nu-\kappa)_n (\tfrac{1}{2}-\nu-\kappa)_n\, t^{-2n} + O(\mathrm{e}^{-\frac{1}{2}t^2}\, t^{2\kappa-2N}) \tag{4.18}$$

for $|t| \to \infty$ in $|\arg t| < \frac{3}{4}\pi$ we find that (4.17) becomes

$$\mathrm{e}^{\frac{1}{2}z^2}\left\{\sum_{n=1}^{N-1} \frac{(-)^n}{n!} (\tfrac{1}{2}+\nu-\kappa)_n (\tfrac{1}{2}-\nu-\kappa)_n \int_{-\delta\xi+\mathrm{i}\eta}^{\infty+\mathrm{i}\eta} \mathrm{e}^{-\frac{1}{2}(t-z)^2}\, t^{\lambda}\,\mathrm{d}t + O(z^{\frac{1}{2}a-\frac{1}{2}+2\kappa-2N})\right\}$$

$$= z^{\frac{1}{2}a-\frac{5}{2}+2\kappa}\, \mathrm{e}^{\frac{1}{2}z^2}\left\{\sum_{n=0}^{N-1} \frac{(-)^n}{n!} (\tfrac{1}{2}+\nu-\kappa)_n (\tfrac{1}{2}-\nu-\kappa)_n\, z^{-2n} \int_{-(1+\delta)\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\left(1+\frac{w}{z}\right)^{\lambda} \mathrm{d}w + O(z^{-2N})\right\}, \tag{4.19}$$

where we have put $t = w+z$ and $\lambda = \frac{1}{2}a-\frac{5}{2}+2\kappa-2n$.

The integral appearing in (4.19) may be written as

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\left(1+\frac{w}{z}\right)^{\lambda} \mathrm{d}w - \int_{(1+\delta)\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\left(1-\frac{w}{z}\right)^{\lambda} \mathrm{d}w. \tag{4.20}$$

On the path of integration of the second integral in (4.20) we have for $\mathrm{Re}\,(z) > 0$ and $|z| > 1$

$$\sin\theta < \{\sin^2\theta + \delta^2\cos^2\theta\}^{\frac{1}{2}} \leqslant \left|1-\frac{w}{z}\right| < 1+w,$$

$$\left|\arg\left(1-\frac{w}{z}\right)\right| < \pi,$$

so that $$\left|\int_{(1+\delta)\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\left(1-\frac{w}{z}\right)^{\lambda} \mathrm{d}w\right| < |\mathrm{cosec}\,\theta|^{|\mathrm{Re}(\lambda)|}\, \mathrm{e}^{\pi|\mathrm{Im}\,(\lambda)|} \int_{(1+\delta)\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\,(1+w)^{|\mathrm{Re}\,(\lambda)|}\, dw$$

$$= O(\mathrm{e}^{-\frac{1}{2}(1+\delta)^2\xi^2}) \tag{4.21}$$

for $\xi \to \infty$. The constant in the O symbol tends to infinity as $\theta \to 0$ so that provided $|z| \to \infty$ in $0 < |\arg z| < \frac{1}{2}\pi$ the integral in (4.20) along the path $[(1+\delta)\,\xi, \infty)$ is asymptotically negligible.

It remains to evaluate $$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}w^2}\left(1+\frac{w}{z}\right)^{\lambda} \mathrm{d}w$$

for large $|z|$ in $0 < |\arg z| < \frac{1}{2}\pi$, where $\lambda = \frac{1}{2}a-\frac{5}{2}+2\kappa-2n$. Expanding $\left(1+\dfrac{w}{z}\right)^{\lambda}$ by means of Taylor's theorem in the form (Hardy 1952, p. 476; Whittaker & Watson 1965, p. 109)

$$\left(1+\frac{w}{z}\right)^{\lambda} = 1+\frac{\lambda}{1!}\frac{w}{z}+\ldots+\frac{\lambda(\lambda-1)\ldots(\lambda-m+1)}{m!}\left(\frac{w}{z}\right)^{m} + R_m(w,z),$$

$$R_m(w,z) = \frac{\lambda(\lambda-1)\ldots(\lambda-m)}{m!}\left(\frac{w}{z}\right)^{m+1}\left(1+\frac{w}{z}\right)^{\lambda}\int_0^1 u^m\left(1+u\frac{w}{z}\right)^{-\lambda-1}\mathrm{d}u,$$

which is valid for arbitrary complex λ and z provided $1+w/z$ is not a negative real number, we find

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2}\left(1+\frac{w}{z}\right)^{\lambda} dw = 2\int_0^{\infty} e^{-\frac{1}{2}w^2}\left\{1+\frac{\lambda(\lambda-1)}{2!}\left(\frac{w}{z}\right)^2+\ldots+\frac{\lambda(\lambda-1)\ldots(\lambda-2M+1)}{(2M)!}\left(\frac{w}{z}\right)^{2M}\right\} dw$$
$$+\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} R_{2M}(w,z)\, dw$$
$$= (2\pi)^{\frac{1}{2}} \sum_{n=0}^{M} \frac{(\frac{1}{2}z^2)^{-m}}{m!}(-\tfrac{1}{2}-\tfrac{1}{2}\lambda)_m(-\tfrac{1}{2}\lambda)_m+\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} R_{2M}(w,z)\, dw. \quad (4.22)$$

Now for real w and $0 \leqslant u \leqslant 1$ we have, assuming $|z| > 1$,

$$\sin\theta \leqslant \left|1+\frac{uw}{z}\right| \leqslant 1+|w|,$$
$$\left|\arg\left(1+\frac{uw}{z}\right)\right| < \pi,$$

where $\theta = \arg z$ and in the latter inequality it is supposed that z is not real. Then

$$|R_{2M}(w,z)| \leqslant \frac{|\lambda(\lambda-1)\ldots(\lambda-2M)|}{(2M)!}\left|\frac{w}{z}\right|^{2M+1}\left|\left(1+\frac{w}{z}\right)^{\lambda}\right|\left|\int_0^1 u^{2M}\left|\left(1+\frac{uw}{z}\right)^{-\lambda-1}\right| du\right.$$
$$< \frac{|\lambda(\lambda-1)\ldots(\lambda-2M)|}{(2M+1)!}\left|\frac{w}{z}\right|^{2M+1}(1+|w|)^{2|\mathrm{Re}\,(\lambda)|}\,|\mathrm{cosec}\,\theta|^{1+2|\mathrm{Re}\,(\lambda)|}\,e^{\pi|\mathrm{Im}\,(\lambda)|}$$
$$= K\left|\frac{w}{z}\right|^{2M+1}(1+|w|)^{2|\mathrm{Re}\,(\lambda)|}\,|\mathrm{cosec}\,\theta|^{1+2|\mathrm{Re}\,(\lambda)|},$$

where K is an assignable constant, so that when $|z| > 1$

$$\left|\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} R_{2M}(w,z)\, dw\right| \leqslant \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2}\,|R_{2M}(w,z)|\, dw$$
$$< \frac{2K|\mathrm{cosec}\,\theta|^{1+2|\mathrm{Re}\,(\lambda)|}}{|z|^{2M+1}}\int_0^{\infty} e^{-\frac{1}{2}w^2} w^{2M+1}(1+w)^{2|\mathrm{Re}\,(\lambda)|}\, dw$$
$$= O(z^{-2M-1}), \quad (4.23)$$

since the integral is convergent and independent of z. The constant in the O symbol in (4.23) contains $|\mathrm{cosec}\,\theta|^{1+2|\mathrm{Re}\,(\lambda)|}$ and consequently tends to infinity as $\theta \to 0$.

From (4.19), (4.21), (4.22) and (4.23) the contribution to (4.1) from the path C_2 as $|z| \to \infty$ in $0 < |\arg z| < \frac{1}{2}\pi$ becomes (considering $M \geqslant N$)

$$(2\pi)^{\frac{1}{2}} z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2}\left\{\sum_{n=0}^{N}\frac{(-)^n}{n!}\frac{(\frac{1}{2}+\nu-\kappa)_n(\frac{1}{2}-\nu-\kappa)_n}{z^{2n}}\right.$$
$$\left.\times\left\{\sum_{m=0}^{M}\frac{(\frac{1}{2}-\frac{1}{2}\lambda)_m(-\frac{1}{2}\lambda)_m}{m!\,(\frac{1}{2}z^2)^m}+O(z^{-2M-1})\right\}+O(z^{-2N-2})\right\}$$
$$= (2\pi)^{\frac{1}{2}} z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2}\left\{\sum_{n=0}^{N}\frac{(-)^n}{n!}\frac{(\frac{1}{2}+\nu-\kappa)_n(\frac{1}{2}-\nu-\kappa)_n}{z^{2n}}\sum_{m=0}^{N-n}\frac{(\frac{1}{2}-\frac{1}{2}\lambda)_m(-\frac{1}{2}\lambda)_m}{m!\,(\frac{1}{2}z^2)^m}+O(z^{-2N-1})\right\},$$
$$(4.24)$$

where the terms corresponding to $m > N-n$ in the inner sum have been included in the order term. Putting $r = n+m$ the double sum in (4.24) may be written

$$\sum_{n=0}^{N} \frac{(-)^n}{n!} \frac{(\frac{1}{2}+\nu-\kappa)_n(\frac{1}{2}-\nu-\kappa)_n}{z^{2n}} \sum_{r=n}^{N} \frac{(\frac{1}{2}-\frac{1}{2}\lambda)_{r-n}(-\frac{1}{2}\lambda)_{r-n}}{(r-n)!}(\tfrac{1}{2}z^2)^{n-r}$$

$$= \sum_{r=0}^{N} (\tfrac{1}{2}z^2)^{-r} \sum_{n=0}^{r} \frac{(-)^n(\frac{1}{2}+\nu-\kappa)_n(\frac{1}{2}-\nu-\kappa)_n}{2^n n!\,(r-n)!}(\tfrac{1}{2}-\tfrac{1}{2}\lambda)_{r-n}(-\tfrac{1}{2}\lambda)_{r-n}$$

$$= \sum_{r=0}^{N} \frac{(2z^2)^{-r}}{r!}(\tfrac{5}{2}-\tfrac{1}{2}a-2\kappa)_{2r} \sum_{n=0}^{r} \frac{(-r)_n}{n!} \frac{(\frac{1}{2}+\nu-\kappa)_n(\frac{1}{2}-\nu-\kappa)_n}{(\frac{5}{4}-\frac{1}{4}a-\kappa)_n(\frac{7}{4}-\frac{1}{4}a-\kappa)_n}(\tfrac{1}{2})^n, \tag{4.25}$$

upon reversal of the order of summation and employing the result $(\alpha+n)_{r-n} = (\alpha)_r/(\alpha)_n$. The inner sum in (4.25) may be expressed as a terminating ${}_3F_2$ generalized hypergeometric function so that as $|z|\to\infty$ in $0 < |\arg z| < \frac{1}{2}\pi$

$$\int_{C_2} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}} W_{\kappa,\nu}(t^2)\, dt \simeq (2\pi)^{\frac{1}{2}} z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} \sum_{n=0}^{\infty} \frac{(2z^2)^{-n}}{n!}(\tfrac{5}{2}-\tfrac{1}{2}a-2\kappa)_{2n}$$

$$\times\, {}_3F_2\left(\begin{matrix}-n,\ \frac{1}{2}+\nu-\kappa,\ \frac{1}{2}-\nu-\kappa;\\ \frac{5}{4}-\frac{1}{4}a-\kappa,\ \frac{7}{4}-\frac{1}{4}a-\kappa;\end{matrix}\ \tfrac{1}{2}\right). \tag{4.26}$$

From (4.14) and (4.26) we therefore obtain the asymptotic expansion of $U_{\kappa,\nu}(a,z)$ for $|z|\to\infty$ in $0 < |\arg z| < \frac{1}{2}\pi$ provided 2ν is non-integral

$$U_{\kappa,\nu}(a,z) \simeq (2\pi)^{\frac{1}{2}} z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} \sum_{n=0}^{\infty} \frac{(2z^2)^{-n}}{n!}(\tfrac{5}{2}-\tfrac{1}{2}a-2\kappa)_{2n}\, {}_3F_2\left(\begin{matrix}-n,\ \frac{1}{2}+\nu-\kappa,\ \frac{1}{2}-\nu-\kappa;\\ \frac{5}{4}-\frac{1}{4}a-\kappa,\ \frac{7}{4}-\frac{1}{4}a-\kappa;\end{matrix}\ \tfrac{1}{2}\right)$$

$$+\frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a+2\nu} \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} \frac{(\frac{1}{2}-\nu-\kappa)_n}{(1-2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}-2\nu+2n)$$

$$\times\, {}_2F_1(-n,\ -n+2\nu;\ \tfrac{1}{2}+\nu+\kappa-n;\ \tfrac{1}{2}) + \frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\kappa)}(-z)^{\frac{1}{2}-\frac{1}{2}a-2\nu}$$

$$\times \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} \frac{(\frac{1}{2}+\nu-\kappa)_n}{(1+2\nu)_n}\Gamma(\tfrac{1}{2}a-\tfrac{1}{2}+2\nu+2n)\, {}_2F_1(-n,\ -n-2\nu;\ \tfrac{1}{2}-\nu+\kappa-n;\ \tfrac{1}{2}), \tag{4.27}$$

where $-z$ is to be interpreted as $e^{\mp\pi i} z$ according as z lies in the upper or lower half plane respectively.

The analysis of the contribution to $U_{\kappa,\nu}(a,z)$ from the path C_2 breaks down as $\arg z\to 0$ since a part of this path then lies in a neighbourhood of the origin for *fixed* large $|z|$ where use of the expansion (4.18) for $W_{\kappa,\nu}(t^2)$ is no longer permitted. When $\arg z = 0$, however, the original path of integration in (2.2) passes over the saddle point at $t = z$ and no deformation of the path $[0,\infty)$ into the paths C_1 and C_2 is necessary. By means of analysis identical to that employed in estimating (4.17) it is found that, when $\arg z = 0$, $U_{\kappa,\nu}(a,z)$ has the exponential asymptotic expansion given by the right hand side of (4.26). This exponential expansion is, moreover, uniformly valid in $\arg z$ as $\arg z\to 0$ since the path of integration in (2.2) may be continuously deformed to pass over the saddle point situated at $t = z$ and the right hand side of (4.26) consequently represents the asymptotic expansion of $U_{\kappa,\nu}(a,z)$ in $|\arg z| < \frac{1}{4}\pi$ in the sense of Poincaré.†

It will be observed from (4.27) that on crossing $\arg z = 0$, where the algebraic expansions are of maximum degree of subdominancy, the coefficients of $z^{\frac{1}{2}-\frac{1}{2}a\pm 2\nu}$ multiplying the algebraic expansions change discontinuously by the factor $\exp[2\pi i(\frac{1}{2}-\frac{1}{2}a\pm 2\nu)]$. This is the Stokes phenomenon

† The asymptotic expansion of $U_{\kappa,\nu}(a,z)$ given in (4.27) is complete in the wider sense of Olver (1964).

where the coefficients of the subdominant algebraic terms change discontinuously in a neighbourhood of $\arg z = 0$ (which may, for convenience, be taken to be $\arg z = 0$) in order to preserve the single-valuedness of $U_{\kappa,\nu}(a,z)$ as z describes a circuit about the origin in the z-plane. On $\arg z = \pm\frac{1}{2}\pi$ the exponential expansion is of maximum degree of subdominancy and its coefficient changes discontinuously† becoming zero in $\mathrm{Re}\,(z) < 0$. The rays $\arg z = 0, \pm\frac{1}{2}\pi$ are consequently Stokes lines for the function defined by $U_{\kappa,\nu}(a,z)$ with the rays $\arg z = \pm\frac{1}{4}\pi$ being anti-Stokes lines upon which the exponential and algebraic expansions are equally significant.

When 2ν takes on integral values, the analysis in §4 (a) of the contribution from the path C_1 needs to be modified. Employing the series representation of $W_{\kappa,\nu}(t^2)$ for integral values of 2ν it can be shown that the contribution to (4.1) from C_1 yields the expansion given in (3.8), as must be the case since the domains of validity of the algebraic expansions resulting from C_1 and that from (3.1) overlap in $\frac{1}{4}\pi < |\arg z| < \frac{1}{2}\pi$. The exponential expansion (4.26) remains unaltered for integral values of 2ν since the expansion of $W_{\kappa,\nu}(t^2)$ in (4.18) is valid for arbitrary ν.

The asymptotic analysis of $U_{\kappa,\nu}(a,z)$ in the half plane $|\arg z| < \frac{1}{2}\pi$ has been obtained by means of the integral representation (2.2) which defines $U_{\kappa,\nu}(a,z)$ only in the domain of the parameters given by $\mathrm{Re}\,(a) > 1+4\,|\mathrm{Re}\,(\nu)|$. This condition arises from the behaviour of the integrand in (2.2) near the origin in the t-plane and consequently only affects the contribution to (4.1) from the path C_1. It will be shown in §5 how the integral representation (2.2) can be continued analytically into $\mathrm{Re}\,(a) \leqslant 1+4\,|\mathrm{Re}\,(\nu)|$, essentially by replacing the divergent part of the integral by a finite number of terms of (4.14), and that the algebraic expansion (4.14) resulting from the path C_1 holds in this wider domain of the parameters.

The special case $\kappa = \frac{1}{2}\pm\nu$ presents some interest since from (2.16) $U_{\kappa,\nu}(a,z)$ may be expressed in terms of the Weber parabolic cylinder function, with (2.16) holding for all values of a and ν, except those in (2.10), by analytic continuation. The exponential expansion (1.3) becomes in this case

$$S^{(1)}_{\frac{1}{2}\pm\nu,\,\nu}(z) = (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2z^2)^{-n}}{n!}\left(\tfrac{3}{2}-\tfrac{1}{2}a\mp 2\nu\right)_{2n},$$

since the ${}_3F_2$ hypergeometric function is equal to unity, and the algebraic expansion in the form (1.5) reduces to

$$S^{(2)}_{\frac{1}{2}\pm\nu,\,\pm\nu}(z) = 0,$$

$$S^{(2)}_{\frac{1}{2}\pm\nu,\,\mp\nu}(z) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!}(2z^2)^{-n}\,\Gamma\left(\tfrac{1}{2}a-\tfrac{1}{2}\pm 2\nu+2n\right),$$

since the ${}_2F_1$ hypergeometric function contracts to ${}_1F_0(-n;;\frac{1}{2}) = 2^{-n}$. Employing the asymptotic expansion of the Weber function for $|z|\to\infty$ (Whittaker & Watson 1965, p. 347) it can be seen that the expansions (1.6) are in agreement in their common ranges of validity with that of the right hand side of (2.16).

5. The extension of $U_{\kappa,\nu}(a,z)$ outside the domain $\mathrm{Re}\,(a) > 1+4|\mathrm{Re}\,(\nu)|$

The integral representation (2.2) of the solution $U_{\kappa,\nu}(a,z)$ of (2.1) is valid only for those values of a and ν which satisfy the inequality (2.4):

$$\mathrm{Re}\,(a) > 1+4\,|\mathrm{Re}\,(\nu)|.$$

† Extension of the analysis in §4 (b) to the domain $|\arg z| \leqslant \frac{1}{2}\pi+\epsilon$ where $\epsilon > 0$, shows in fact that the coefficient multiplying the exponential expansion changes in two equal half-jumps as the rays $\arg z = \pm\frac{1}{2}\pi$ are crossed.

These values correspond to the domain D_0 in figure 2. Since ν is defined via (2.3) from the constants a and b occurring in the differential equation (2.1), this represents an awkward restriction on the class of equations which may be treated. For instance, if $a = 1$, then the integral representation solution fails to exist for any b. We therefore show in this section how the function given in D_0 by the integral representation (2.2) may be extended, first to the domain D_1 given by

$$\mathrm{Re}\,(a) + 3 > 4|\mathrm{Re}\,(\nu)|, \tag{5.1}$$

and thence to the domain D_n, for arbitrary positive integer n, given by

$$\mathrm{Re}\,(a) + 4n - 1 > 4\,|\mathrm{Re}\,(\nu)|. \tag{5.2}$$

Since for every pair (ν, a) we have $(\mathrm{Re}\,(\nu), \mathrm{Re}\,(a)) \in D_n$ for some integer n, (5.2) provides an extension to all finite values of (ν, a). It can be shown by standard methods (Wood 1968) that these extensions remain solutions of the differential equation (2.1). The removal of the restriction (2.4) is particularly important in applications where b is an eigenvalue parameter talking all complex values (see Wood 1971).

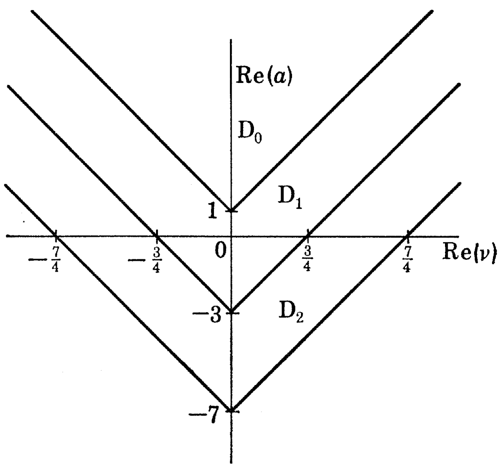

FIGURE 2. The domains $D_0, D_1, D_2, \ldots$ in the $(\mathrm{Re}\,(\nu), \mathrm{Re}\,(a))$-plane defined by $\mathrm{Re}\,(a) + 4n - 1 > 4|\mathrm{Re}\,(\nu)|$ for $n = 0, 1, 2, \ldots$ employed in the successive analytic continuations of $U_{\kappa,\nu}(a, z)$.

For $\mathrm{Re}\,(a) > 1$ let Δ_0 be the infinite strip in the ν-plane given by (2.4): this corresponds to the domain D_0 in the $(\mathrm{Re}\,(a), \mathrm{Re}\,(\nu))$-plane. $U_{\kappa,\nu}(a, z)$ is a meromorphic function of ν in Δ_0. We begin by continuing $U_{\kappa,\nu}(a, z)$ analytically from the strip Δ_0 to the larger strip Δ_1 given by (5.1) and corresponding to D_1. From (2.2) we have

$$U_{\kappa,\nu}(a, z) = \int_0^1 e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, W_{\kappa,\nu}(t^2)\, dt + \int_1^\infty e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, W_{\kappa,\nu}(t^2)\, dt.$$

There is no difficulty over the convergence of the second integral which is a meromorphic function of ν in the whole plane, but the first integral has a singularity at its lower end point if $\nu \notin \Delta_0$. Using the series for the Whittaker function (Slater 1960, p. 14) we may expand the first integrand as follows

$$e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, W_{\kappa,\nu}(t^2) = e^{t(z-\frac{1}{2}t)}\{T_{\kappa,\nu}(t) + T_{\kappa,-\nu}(t)\}, \tag{5.3}$$

where
$$T_{\kappa,\nu}(t) = \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\kappa)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}-\nu-\kappa)_m\, t^{2m-\frac{3}{2}-2\nu+\frac{1}{2}a}}{(1-2\nu)_m\, m!}$$

and the series are uniformly and absolutely convergent over any finite t-interval. Thus we may integrate term-by-term to obtain

$$\int_0^1 e^{zt}\, t^{\frac{1}{2}a-\frac{5}{4}}\, W_{\kappa,\nu}(t^2)\,dt = \sum_{m=0}^{\infty} \alpha_m(\kappa,\nu)\, I_{m,\nu}(z) + \sum_{m=0}^{\infty} \alpha_m(\kappa,-\nu)\, I_{m,-\nu}(z),$$

where $$\alpha_m(\kappa,\nu) = \frac{\Gamma(2\nu)\,(\frac{1}{2}-\nu-\kappa)_m}{\Gamma(\frac{1}{2}+\nu-\kappa)\,(1-2\nu)_m\, m!}, \quad I_{m,\nu}(z) = \int_0^1 e^{t(z-\frac{1}{2}t)}\, t^{2m-\frac{3}{2}-2\nu+\frac{1}{2}a}\,dt. \tag{5.4}$$

For $\nu \in \Delta_0$, the integrals $I_{m,\pm\nu}(z)$ exist for all m, but for $\nu \in \Delta_n$ these integrals may diverge for $m < n$.

If we put $\mu = \frac{1}{2}a - \frac{1}{2} - 2\nu$ and, for $\nu \in \Delta_0$,

$$I_\mu(z) \equiv I_{0,\nu}(z) = \int_0^1 e^{t(z-\frac{1}{2}t)}\, t^{\mu-1}\,dt,$$

it is easily established that

$$I_\mu(z) = \{e^{z-\frac{1}{2}} - zI_{\mu+1}(z) + I_{\mu+2}(z)\}/\mu, \tag{5.5}$$

and hence that

$$I_\mu(z) = \{(\mu+1-z)\,e^{z-\frac{1}{2}} + (\mu+1+z^2)\, I_{\mu+2}(z) - zI_{\mu+3}(z)\}/\mu(\mu+1). \tag{5.6}$$

But the right hand side of (5.6) is convergent for $\nu \in \Delta_1$ and equal to $I_\mu(z)$ for $\nu \in \Delta_0$. Hence it represents the continuation of $I_{0,\nu}(z)$ to Δ_1 as a meromorphic function of ν with poles at $\nu = \frac{1}{4}a \mp \frac{1}{4}$.

It is now clear that for $m < n$ we can continue the functions represented by $I_{m,\nu}(z)$ into the domain Δ_n by using the result obtained by repeated application of (5.5),

$$I_\mu(z) = \frac{\Gamma(\mu)}{\Gamma(\mu+r+1)}\{P_r(z)\, e^{z-\frac{1}{2}} + Q_{r+1}(z)\, I_{\mu+r+1}(z) + Q_r(z)\, I_{\mu+r+2}(z)\},$$

where P_r and Q_r are polynomials in z of degree r given by the recurrence relations

$$P_r(z) = (\mu+r)\,P_{r-1}(z) + Q_r(z),$$
$$Q_{r+1}(z) = (\mu+r)\,Q_{r-1}(z) - zQ_r(z),$$

with initial values $P_0(z) = Q_0(z) = 1$, $Q_1(z) = -z$.

The continuation of $\sum_{m=0}^{n-1} \alpha_m(\kappa,\nu)\, I_{m,\nu}(z)$ to Δ_n is then given by

$$J_{\kappa,\nu}(z) = \sum_{m=0}^{n-1} \alpha_m(\kappa,\nu)\, \frac{\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2m)}{\Gamma(\frac{1}{2}a-\frac{1}{2}-2\nu+2n)}\{P_{2n-2m-1}(z)\, e^{z-\frac{1}{2}}$$
$$+ Q_{2n-2m}(z)\, I_{\mu+2n}(z) + Q_{2n-2m-1}(z)\, I_{\mu+2n+1}(z)\}. \tag{5.7}$$

$J_{\nu,\kappa}(z)$ is a meromorphic function of ν in Δ_n with simple poles at

$$\nu = \tfrac{1}{4}a - \tfrac{1}{4},\ \tfrac{1}{4}a + \tfrac{1}{4},\ \ldots,\ \tfrac{1}{4}a + n - \tfrac{3}{4}.$$

Thus the analytic continuation of $U_{\kappa,\nu}(z)$ from Δ_0 to Δ_n is provided by the function $U^{(n)}_{\kappa,\nu}(z)$ given by

$$J_{\kappa,\nu}(z) + J_{\kappa,-\nu}(z) + \sum_{m=n}^{\infty} \{\alpha_m(\kappa,\nu)\, I_{m,\nu}(z) + \alpha_m(\kappa,-\nu)\, I_{m,-\nu}(z)\} + \int_1^\infty e^{zt}\, t^{\frac{1}{2}a-\frac{5}{4}}\, W_{\kappa,\nu}(t^2)\,dt, \tag{5.8}$$

where $J_{\kappa,\nu}(z)$ is given by (5.7) and $\alpha_m(\kappa,\nu)$, $I_{m,\nu}(z)$ by (5.4). Notice that $U^{(n)}_{\kappa,\nu}(a,z)$ has poles at the first $2n$ values defined by (2.10). These may be removed by starting from the solution $\mathfrak{U}_{\kappa,\nu}(a,z)$ defined as a multiple of $U_{\kappa,\nu}(a,z)$ by (2.20).

Although we can show that $U^{(n)}_{\kappa,\nu}(a,z)$ is a solution of (2.1) for those values of (a,ν) which satisfy (5.2), it is not in a suitable form for obtaining an asymptotic expansion for large $|z|$, since $J_{\kappa,\pm\nu}(z)$ is arranged in ascending powers of z. We may overcome this difficulty by expanding the term $e^{-\frac{1}{2}t^2}$ in the integrals $I_{m,\pm\nu}(z)$ to obtain a series of integrals of form

$$\int_0^1 e^{zt}\, t^s\, dt = \int_0^\infty e^{zt}\, t^s\, dt - \int_1^\infty e^{zt}\, t^s\, dt. \tag{5.9}$$

For Re $(z) < 0$ the first integral on the left hand side can be written as an inverse power of z multiplied by a gamma function, while the second is small as $|z| \to \infty$.

Alternatively, the expansion of $W_{\kappa,\nu}(t^2)$ in the form (4.7) may be used instead of (5.3) to derive another form of the analytic continuation that is more useful for asymptotic work. With $M_{\kappa,\nu}(t^2)$ defined as in (4.6), $A_n(\nu)$ as in (4.8) and $\Psi_{N,\nu}$ as in (4.9) we may write, for $\nu \in D_0$,

$$\int_{C_1} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, M_{\kappa,-\nu}(t^2)\, dt = e^{i\mu(\pm\pi-\theta)} \sum_{n=0}^{N-1} A_n(\nu)\, e^{-2ni\theta} \int_0^x e^{-xu}\, u^{\mu-1+2n}\, du - e^{-i\theta} \int_0^x e^{-xu}\, \Psi_{N,\nu}(t)\, du, \tag{4.9}$$

where on the path C_1, $t = u \exp i(\pm\pi-\theta)$ with the upper or lower sign chosen as in §4 (*a*). The formula (5.9) may then be used to show that the first term on the right hand side of (4.9) corresponds to the first N terms of the first half of the expansion (4.14) which are clearly analytic for $\nu \in D_N$, apart from the poles given by (2.10). Since the second term in (4.9) is clearly analytic in D_N it follows that the expansion and this remainder term give the analytic continuation of the left hand side of (4.9) from D_0 to D_N. By repeating the argument with ν for $-\nu$ and combining the two results using (4.5) we obtain the continuation of $\int_{C_1} e^{zt}\, t^{\frac{1}{2}a-\frac{5}{2}}\, W_{\kappa,\nu}(t^2)\, dt$. Since there is no problem over the analyticity of the integral along the path C_2, it now follows from (4.1) that for $|\arg z| < \frac{1}{2}\pi$ the solution $U_{\kappa,\nu}(a,z)$ may be continued analytically from D_0 to D_N and that the analytic continuation has the same asymptotic expansion as the original integral representation. The asymptotic expansion (4.27) of the solution $U_{\kappa,\nu}(a,z)$ is then valid for all values of the parameters a and ν apart from (2.10). Since the expansion (3.7) of $U_{\kappa,\nu}(a,z)$ in $|\arg(-z)| < \frac{3}{4}\pi$ was obtained for all values of the parameters except (2.10) and integral values of 2ν, the full range expansions (1.6) are consequently seen to hold, apart from these exceptions, for all values of a, ν and κ.

6. Exponentially small solutions

The solution $U_{\kappa,\nu}(a,z)$ of the differential equation (2.1) for $|z| \to \infty$ is exponentially large in $|\arg z| < \frac{1}{4}\pi$ and algebraic in character in $|\arg(-z)| < \frac{3}{4}\pi$. In many physical applications, however, it is necessary to have solutions which are exponentially small in some sector for $|z| \to \infty$. In this section we consider linear combinations of the solutions in (2.18) which have an exponentially small expansion in a certain sector of the z-plane by arranging for the algebraic expansions in a domain containing this sector to cancel.

Consider the solution $U_1(z)$ defined by

$$U_1(z) = U_{-\kappa,\nu}(a, iz) + \alpha U_{\kappa,\nu}(a, -z) + \beta U_{-\kappa,\nu}(a, -iz), \tag{6.1}$$

where α and β are constants to be chosen. Employing the integral representation (2.2) for each of the solutions on the right hand side of (6.1), we deform the paths of integration in the t-plane in the manner described in §4 into the paths $C_1^{(i)}$ and $C_2^{(i)}$ $(i = 1, 2, 3)$ as shown in figure 3. Along the paths $C_1^{(i)}$ the phase of t is chosen so that the argument of the exponential factor in each integrand is real and negative. The contribution to $U_1(z)$ resulting from the paths $C_1^{(i)}$ yields the algebraic expansion, and for $z = x\,e^{i\theta}$ lying in $\mathrm{Re}\,(z) > 0$ may be written

$$e^{i(\frac{1}{2}\pi-\theta)\,\gamma}\int_0^x e^{-ux}\,u^{\gamma-1}\{W_{-\kappa,\,\nu}(\tau^2\,e^{\pi i}) + \alpha\,e^{-\frac{1}{2}\pi i\gamma}\,W_{\kappa,\,\nu}(\tau^2) + \beta\,e^{-\pi i\gamma}\,W_{-\kappa,\,\nu}(\tau^2\,e^{-\pi i})\}\,du, \tag{6.2}$$

where $\gamma = \frac{1}{2}a - \frac{3}{2}$ and $\tau = u\,e^{-i\theta}$. Making use of the analytic continuation formulae (Slater 1960, p. 28)

$$W_{-\kappa,\,\nu}(\tau^2\,e^{\pm\pi i}) = \frac{\pi}{\sin 2\pi\nu}\left\{\frac{-\,e^{\pm\pi i(\frac{1}{2}+\nu)}\,M_{\kappa,\,\nu}(\tau^2)}{\Gamma(\frac{1}{2}-\nu+\kappa)\,\Gamma(1+2\nu)} + \frac{e^{\pm\pi i(\frac{1}{2}-\nu)}\,M_{\kappa,\,-\nu}(\tau^2)}{\Gamma(\frac{1}{2}+\nu+\kappa)\,\Gamma(1-2\nu)}\right\},$$

$$W_{\kappa,\,\nu}(\tau^2\,e^{\pm 2\pi i}) = \frac{\pi}{\sin 2\pi\nu}\left\{\frac{e^{\pm 2\pi i\nu}\,M_{\kappa,\,\nu}(\tau^2)}{\Gamma(\frac{1}{2}-\nu-\kappa)\,\Gamma(1+2\nu)} - \frac{e^{\mp 2\pi i\nu}\,M_{\kappa,\,-\nu}(\tau^2)}{\Gamma(\frac{1}{2}+\nu-\kappa)\,\Gamma(1-2\nu)}\right\},$$

together with (4.5), we can show that the integrand in (6.2) vanishes identically if the constants α and β are chosen to satisfy

$$\alpha = \frac{2\pi\,e^{\frac{1}{2}\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)}}{\Gamma(\frac{1}{2}+\nu+\kappa)\,\Gamma(\frac{1}{2}-\nu+\kappa)}, \quad \beta = e^{\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)}. \tag{6.3}$$

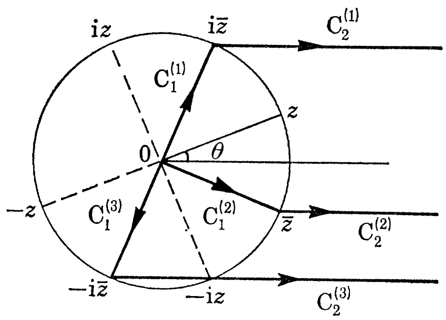

FIGURE 3. The paths $C_1^{(i)}$ and $C_2^{(i)}$ $(i = 1, 2, 3)$ in the t-plane for the linear combination $U_1(z)$ when $\mathrm{Re}\,(z) > 0$. Along the paths $C_1^{(i)}$ the phase of t is such that zt is real and negative. The paths $C_2^{(i)}$ pass to infinity parallel to the positive real axis.

It now follows with these values of α and β that for $|z| \to \infty$ in $\mathrm{Re}\,(z) > 0$ $U_1(z)$ is purely exponential in character possessing no algebraic expansion. This cancellation of the algebraic expansions no longer holds when z lies in $\mathrm{Re}\,(z) < 0$ on account of the discontinuous change in the coefficients of the algebraic expansions associated with the Stokes phenomenon as one or more of the points $-z$, $\pm iz$ cross the positive real axis. The values of the constants α and β in (6.3) have been obtained only for those values of a and ν satisfying (2.4) corresponding to the domain of validity of the integral representation (2.2). This result may be readily extablished for all parameter values (except those in (2.10)) by direct substitution of the expansions (1.6) into (6.1) and (6.3) and making use of the fact that

$$S^{(2)}_{-\kappa,\,\pm\nu}(iz) = \frac{\Gamma(\frac{1}{2}\pm\nu-\kappa)}{\Gamma(\frac{1}{2}\pm\nu+\kappa)}\,S^{(2)}_{\kappa,\,\pm\nu}(z) \tag{6.4}$$

obtained from (1.4) and (3.4).

The asymptotic expansion for $|z| \to \infty$ of $U_1(z)$ in the sense of Poincaré is consequently given by

$$U_1(z) \simeq (e^{\frac{1}{2}\pi i} z)^{\frac{1}{2}a-\frac{5}{2}-2\kappa} e^{-\frac{1}{2}z^2} S^{(1)}_{-\kappa,\nu}(iz), \qquad |\arg z| < \tfrac{3}{4}\pi, \tag{6.5}$$

$$U_1(z) \simeq \frac{2\pi e^{\frac{1}{2}\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)}}{\Gamma(\frac{1}{2}+\nu+\kappa)\,\Gamma(\frac{1}{2}-\nu+\kappa)} (-z)^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} S^{(1)}_{\kappa,\nu}(z), \quad |\arg(-z)| < \tfrac{1}{4}\pi. \tag{6.6}$$

The solution $U_1(z)$ is therefore seen to be exponentially small in the sector $|\arg z| < \frac{1}{4}\pi$, being exponentially large in the rest of the z-plane except on the rays $\arg z = \pm\frac{1}{4}\pi$, $\arg z = \pm\frac{3}{4}\pi$. On the anti-Stokes lines $\arg z = \pm\frac{3}{4}\pi$ the asymptotic expansion of $U_1(z)$ is the sum of (6.5) and (6.6) together with the algebraic expansions

$$-2i\, e^{\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)} (e^{-\frac{1}{2}\pi i} z)^{\frac{1}{2}-\frac{1}{2}a} \{\sin(\tfrac{1}{2}-\tfrac{1}{2}a+2\nu)\,\pi (e^{-\frac{1}{2}\pi i} z)^{2\nu} S^{(2)}_{-\kappa,\nu}(iz) + \sin(\tfrac{1}{2}-\tfrac{1}{2}a-2\nu)\,\pi (e^{-\frac{1}{2}\pi i} z)^{-2\nu} S^{(2)}_{-\kappa,-\nu}(iz)\} \tag{6.7}$$

on $\arg z = \frac{3}{4}\pi$, and

$$2i(e^{\frac{1}{2}\pi i} z)^{\frac{1}{2}-\frac{1}{2}a} \{\sin(\tfrac{1}{2}-\tfrac{1}{2}a+2\nu)\,\pi (e^{\frac{1}{2}\pi i} z)^{2\nu} S^{(2)}_{-\kappa,\nu}(iz) + \sin(\tfrac{1}{2}-\tfrac{1}{2}a-2\nu)\,\pi (e^{\frac{1}{2}\pi i} z)^{-2\nu} S^{(2)}_{-\kappa,-\nu}(iz)\} \tag{6.8}$$

on $\arg z = -\frac{3}{4}\pi$. It is to be noted that the algebraic expansions (6.7) and (6.8) hold in the wider sense of Olver (1964) in the domains $\frac{1}{2}\pi < \arg z < \pi$ and $-\pi < \arg z < -\frac{1}{2}\pi$ respectively.

A second exponentially small solution may be similarly shown to be

$$U_2(z) = U_{\kappa,\nu}(a, z) + \frac{2\pi e^{\frac{1}{2}\pi i(\frac{1}{2}a-\frac{5}{2}+2\kappa)}}{\Gamma(\frac{1}{2}+\nu-\kappa)\,\Gamma(\frac{1}{2}-\nu-\kappa)} U_{-\kappa,\nu}(a, iz) + e^{\pi i(\frac{1}{2}a-\frac{5}{2}+2\kappa)} U_{\kappa,\nu}(a, -z), \tag{6.9}$$

possessing the asymptotic expansion for $|z| \to \infty$ in the sense of Poincaré

$$U_2(z) \simeq z^{\frac{1}{2}a-\frac{5}{2}+2\kappa} e^{\frac{1}{2}z^2} S^{(1)}_{\kappa,\nu}(z), \qquad |\arg(iz)| < \tfrac{3}{4}\pi, \tag{6.10}$$

$$U_2(z) \simeq \frac{2\pi e^{\pi i(\frac{1}{2}a-\frac{5}{2})}}{\Gamma(\frac{1}{2}+\nu-\kappa)\,\Gamma(\frac{1}{2}-\nu-\kappa)} z^{\frac{1}{2}a-\frac{5}{2}-2\kappa} e^{-\frac{1}{2}z^2} S^{(1)}_{-\kappa,\nu}(iz), \quad |\arg(-iz)| < \tfrac{1}{4}\pi. \tag{6.11}$$

This solution is exponentially small in $|\arg(iz)| < \frac{1}{4}\pi$. On the anti-Stokes lines $\arg z = \frac{5}{4}\pi$ and $\arg z = -\frac{1}{4}\pi$ the asymptotic expansion of $U_2(z)$ is given by the sum of (6.10) and (6.11) together with the algebraic expansions

$$-2i\, e^{\pi i(\frac{1}{2}a-\frac{5}{2}+2\kappa)} (e^{-\pi i} z)^{\frac{1}{2}-\frac{1}{2}a} \{\sin(\tfrac{1}{2}-\tfrac{1}{2}a+2\nu)\,\pi (e^{-\pi i} z)^{2\nu} S^{(2)}_{\kappa,\nu}(z) + \sin(\tfrac{1}{2}-\tfrac{1}{2}a-2\nu)\,\pi (e^{-\pi i} z)^{-2\nu} S^{(2)}_{\kappa,-\nu}(z)\} \tag{6.12}$$

on $\arg z = \frac{5}{4}\pi$, and

$$2iz^{\frac{1}{2}-\frac{1}{2}a}\{\sin(\tfrac{1}{2}-\tfrac{1}{2}a+2\nu)\,\pi z^{2\nu} S^{(2)}_{\kappa,\nu}(z) + \sin(\tfrac{1}{2}-\tfrac{1}{2}a-2\nu)\,\pi z^{-2\nu} S^{(2)}_{\kappa,-\nu}(z)\} \tag{6.13}$$

on $\arg z = -\frac{1}{4}\pi$. The algebraic expansions (6.12) and (6.13) hold in the wider sense of Olver in the domains $\pi < \arg z < \frac{3}{2}\pi$ and $-\frac{1}{2}\pi < \arg z < 0$ respectively.

The two further linear combinations of the solutions in (2.18) which possess domains in which the asymptotic behaviour is exponentially small, may be shown to be given by

$$\begin{aligned} U_3(z) &= U_{\kappa,\nu}(a, -z) + \frac{2\pi\, e^{\frac{1}{2}\pi i(\frac{1}{2}a-\frac{5}{2}+2\kappa)}}{\Gamma(\frac{1}{2}+\nu-\kappa)\,\Gamma(\frac{1}{2}-\nu-\kappa)} U_{-\kappa,\nu}(a, -iz) + e^{\pi i(\frac{1}{2}a-\frac{5}{2}+2\kappa)} U_{\kappa,\nu}(a, z) \\ &= U_2(-z), \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} U_4(z) &= U_{-\kappa,\nu}(a, -iz) + \frac{2\pi\, e^{\frac{1}{2}\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)}}{\Gamma(\frac{1}{2}+\nu+\kappa)\,\Gamma(\frac{1}{2}-\nu+\kappa)} U_{\kappa,\nu}(a, z) + e^{\pi i(\frac{1}{2}a-\frac{5}{2}-2\kappa)} U_{-\kappa,\nu}(a, iz) \\ &= U_1(-z). \end{aligned} \tag{6.15}$$

Apart from values of the parameters satisfying (2.10), the solutions $U_1(z)$, $U_2(z)$, $U_3(z)$ and $U_4(z)$ constitute a fundamental system of solutions of the differential equation (2.1) possessing domains in which the asymptotic behaviour as $|z| \to \infty$ is exponentially damped.

References

Abramowitz, M. & Stegun, I. (ed.) 1965 *Handbook of mathematical functions*. New York: Dover.

Copson, E. T. 1935 *The theory of functions of a complex variable*. Oxford University Press.

Erokhin, N. S. 1971 *Differen. Urav.* **7**, 970–980. [English translation: *Diffl. Equat.* **7**, 735–742 (1971).]

Fidone, I. & Paris, R. B. 1974 *Phys. Fluids* **17**, 1921–1922.

Hardy, G. H. 1952 *Pure Mathematics*. Cambridge University Press.

Lakin, W. D. 1974 *Q. appl. Math.* **32**, 11–27.

Lakin, W. D. & Nachman, A. 1978 *Q. appl. Math.* **35**, 479–493.

Langer, R. E. 1957 *Trans. Am. math. Soc.* **84**, 144–191.

Langer, R. E. 1959 *Trans. Am. math. Soc.* **92**, 371–410.

Lin, C. C. & Rabenstein, A. L. 1960 *Trans. Am. math. Soc.* **94**, 24–57.

Meijer, C. S. 1935 *Q. Jl. Math.* **6**, 241–248.

Olver, F. W. J. 1964 In *Asymptotic solutions of differential equations and their applications* (ed. C. H. Wilcox), pp. 163–183. New York: Wiley.

Olver, F. W. J. 1974 *Asymptotics and special functions*. New York: Academic Press.

Paris, R. B. 1975 *Proc. R. Soc. Lond.* A **346**, 171–207.

Paris, R. B. 1978 *Association Euratom-CEA report EUR-CEA FC* 940 (to be published in *Proc. R. Soc. Edinb.*).

Slater, L. J. 1960 *Confluent hypergeometric functions*. Cambridge University Press.

Slater, L. J. 1966 *Generalized hypergeometric functions*. Cambridge University Press.

Watson, G. N. 1944 *Theory of Bessel Functions*. Cambridge University Press.

Whittaker, E. T. & Watson, G. N. 1965 *Modern Analysis*. Cambridge University Press.

Wood, A. D. 1968 Asymptotic methods for the determination of deficiency indices. Ph.D. dissertation, University of St Andrews.

Wood, A. D. 1971 *J. Lond. math. Soc.* II, **3**, 96–100.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES